# Well-Posedness of the Einstein-Euler System in Asymptotically Flat Spacetimes: The Evolution Equations

Uwe Brauer and Lavi Karp

**Departamento Matemática Aplicada**
**Universidad Complutense Madrid**
**28040 Madrid, Spain**
*E-mail:* oub@mat.ucm.es

**Department of Mathematics**
**ORT Braude Collegekarp@braude.ac.il**
**P. O. Box 78, 21982 Karmiel, Israel**
*E-mail:* karp@braude.ac.il

**Abstract**

This is the second part of our work concerning the well posedness of the coupled Einstein–Euler system in an asymptotically flat spacetime. Here we prove a local in time existence and uniqueness theorem of classical solutions of the evolution equations. We use the condition that the energy density might vanish or tends to zero at infinity and that the pressure is a certain function of the energy density, conditions which are used to describe simplified stellar models. In order to achieve our goals we are enforced, by the complexity of the problem, to deal with these equations in a new type of weighted Sobolev spaces of fractional order. Beside their construction, we develop tools for PDEs and techniques for hyperbolic equations in these spaces. The well posedness is obtained in these spaces. The results obtained are related to and generalize earlier works of Rendall [23] for the Euler-Einstein system under the restriction of time symmetry and of Gamblin [10] for the simpler Euler–Poisson system.

## 1 Introduction

This paper deals with the Cauchy problem for the Einstein-Euler system describing a relativistic self-gravitating perfect fluid, whose density either has, compact support or falls off at infinity in an appropriate manner, that is, the density belongs to a certain weighted Sobolev space.

The evolution of the gravitational field is described by the Einstein equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi T_{\alpha\beta} \tag{1.1}$$

where $g_{\alpha\beta}$ is a semi Riemannian metric having a signature $(-,+,+,+)$, $R_{\alpha\beta}$ is the Ricci curvature tensor, these are functions of $g_{\alpha\beta}$ and its first and second order partial derivatives and $R$ is the scalar curvature. The right hand side of (1.1) consists of the energy-momentum tensor of the matter, $T_{\alpha\beta}$ and in the case of a perfect fluid the latter takes the form

$$T^{\alpha\beta} = (\epsilon + p) u^\alpha u^\beta + p g^{\alpha\beta}, \tag{1.2}$$

where $\epsilon$ is the energy density, $p$ is the pressure and $u^\alpha$ is the four-velocity vector. The vector $u^\alpha$ is a unit timelike vector, which means that it is required to satisfy the normalization condition

$$g_{\alpha\beta} u^\alpha u^\beta = -1. \tag{1.3}$$

The Euler equations describing the evolution of the fluid take the form

$$\nabla_\alpha T^{\alpha\beta} = 0, \tag{1.4}$$

where $\nabla$ denotes the covariant derivative associated to the metric $g_{\alpha\beta}$. Equations (1.1) and (1.4) are not sufficient to determinate the structure uniquely, a functional relation between the pressure $p$ and the energy density $\epsilon$ (equation of state) is also necessary. We choose an equation of state that has been used in astrophysical problems. It is the analogue of the well known polytropic equation of state in the non-relativistic theory, given by

$$p = f(\epsilon) = K\epsilon^{\gamma}, \qquad K, \gamma \in \mathbb{R}^{+}, \qquad 1 < \gamma. \tag{1.5}$$

The sound velocity is denoted by

$$\sigma^2 = \frac{\partial p}{\partial \epsilon}. \tag{1.6}$$

The unknowns of these equations are the semi Riemannian metric $g_{\alpha\beta}$, the velocity vector $u^{\alpha}$ and the energy density $\epsilon$. These are functions of $t$ and $x^a$ where $x^a$ $(a = 1, 2, 3)$ are the Cartesian coordinates on $\mathbb{R}^3$. The alternative notation $x^0 = t$ will also be used and Greek indices will take the values $0, 1, 2, 3$ in the following.

The common method to solve the Cauchy problem for the Einstein equations consists usually of two steps. Unlike ordinary initial value problems, initial data must satisfy constraint equations intrinsic to the initial hypersurface. Therefore, the first step is to construct solutions of these constraints. The second step is to solve the evolution equations with these initial data, in the present case these are first order symmetric hyperbolic systems. As we describe later in detail, the complexity of our problems forces us to consider an additional third step, that is, after solving the constraint equations, we have to construct the initial data for the fluid equations.

The nature of this Einstein-Euler system (1.1), (1.4) and (1.5) forces us to treat both the constraint and the evolution equations in the same type of functional spaces. Under the above consideration, we have established the well posedness of this Einstein-Euler system in weighted Sobolev spaces of fractional order. Oliynyk has recently studied the Newtonian limit of this system in weighted Sobolev spaces of integer order [22]

We will briefly resume the situation in the mathematical theory of self gravitation perfect fluids describing compact bodies, such as stars: For the Euler-Poisson system Makino proved a local existence theorem in the case the density has compact support and it vanishes at the boundary, [19]. Since the Euler equations are singular when the density $\rho$ is zero, Makino had to regularize the system by introducing a new matter variable $(w = M(\rho))$. His solution however, has some disadvantages such as the fact they do not contain static solutions and moreover, the connection between the physical density and the new matter density remains obscure.

Rendall generalized Makino's result to the relativistic case of the Einstein–Euler equations, [23]. His result however suffers from the same disadvantages as Makino's result and moreover it has two essential restrictions: 1. Rendall assumed time symmetry, that means that the extrinsic curvature of the initial manifold is zero and therefore the Einstein's constraint equations are reduced to a single scalar equation; 2. Both the data and solutions are $C_0^{\infty}$ functions. This regularity condition implies a severe restriction on the equation of state $p = K\epsilon^{\gamma}$, namely $\gamma \in \mathbb{N}$.

Similarly to Makino and Rendall, we have also used the Makino variable

$$w = M(\epsilon) = \epsilon^{\frac{\gamma-1}{2}}. \tag{1.7}$$

Our approach is motivated by the following observation. As it turns out, the system of evolution equations have the following form

$$A^0 \partial_t U + \sum_{k=1}^{3} A^k \partial_k U = Q(\epsilon, ..), \tag{1.8}$$

where the unknown $U$ consists of the gravitational field $g_{\alpha\beta}$ the velocity of the fluid $u^{\alpha}$ and the Makino variable $w$, and the lower order term $Q$ contains the energy density $\epsilon$. Thus, we need to estimate $\epsilon$ by $w$ in the corresponding norm of the function spaces. Combining this estimation with the Makino variable (1.7), it results in an algebraic relation between the order of the functional space $k$ and the coefficient $\gamma$ of the equation of state (1.5) of the form

$$1 < \gamma \leq \frac{2+k}{k}. \tag{1.9}$$

This relation can be easily derived by considering $\|\partial^\alpha w\|_{L_2}$, $|\alpha| \leq k$. Moreover, it can be interpreted either as a restriction on $\gamma$ or on $k$. Thus, unlike typical hyperbolic systems where often the regularity parameter is bounded from below, here we have both lower and upper bounds for differentiability conditions of the sort $\frac{5}{2} < k \leq \frac{2}{\gamma-1}$. A similar phenomenon for the Euler-Poisson equations was noted by Gamblin [10].

We want to interpret (1.9) as a restriction on $k$ rather than on $\gamma$. Therefore, instead of imposing conditions on the equation of state and in order to sharpen the regularity conditions for existence theorems, we are lead to the conclusion of considering function spaces of fractional order, and in addition, the Einstein equations consist of quasi linear hyperbolic and elliptic equations. The only function spaces which are known to be useful for existence theorems of the constraint equations in the asymptotically flat case, are the weighted Sobolev spaces $H_{k,\delta}$, $k \in \mathbb{N}$, $\delta \in \mathbb{R}$, which were introduced by Nirenberg and Walker, [21] and Cantor [6], and they are the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$(\|u\|_{k,\delta})^2 = \sum_{|\alpha|\leq k} \int \left((1+|x|)^{\delta+|\alpha|}|\partial^\alpha u|\right)^2 dx. \tag{1.10}$$

Hence we are forced to consider new function spaces $H_{s,\delta}$, $s \in \mathbb{R}$ which generalize the spaces $H_{k,\delta}$ to fractional order. The well posedness of the Einstein-Euler system is obtained in these spaces. In order to achieve this, we have to solve both the constraint and the evolution equations in the $H_{s,\delta}$ spaces.

Another difficulty which arises from the non-linear equation of state (1.5) is the compatibility problem of the initial data for the fluid and the gravitational field. There are three types of initial data for the Einstein-Euler system:

- The gravitational data is a triple $(M, h, K)$, where $M$ is space-like manifold, $h = h_{ab}$ is a proper Riemannian metric on $M$ and $K = K_{ab}$ is a second fundamental form on $M$ (extrinsic curvature). The pair $(h, K)$ must satisfy the constrain equations

$$\left\{ \begin{array}{rcl} R(h) - K_{ab}K^{ab} + (h^{ab}K_{ab})^2 & = & 16\pi z, \\ {}^{(3)}\nabla_b K^{ab} - {}^{(3)}\nabla^b(h^{bc}K_{bc}) & = & -8\pi j^a, \end{array} \right. \tag{1.11}$$

  where $R(h) = h^{ab}R_{ab}$ is the scalar curvature with respect to the metric $h$.

- The matter variables, consisting of the energy density $z$ and the momentum density $j^a$, appear in the right hand side of the constraints (1.11).

- The initial data for Makino's variable $w$ and the velocity vector $u^\alpha$ of the perfect fluid.

Letting $\bar{u}^\alpha$ denote the projection of the velocity vector $u^\alpha$ on the tangent space of the initial manifold $M$, leads to the following relations

$$\left\{ \begin{array}{rcl} z & = & \epsilon + (\epsilon + p)h_{ab}\bar{u}^a\bar{u}^b \\ j^\alpha & = & (\epsilon + p)\bar{u}^a\sqrt{1 + h_{ab}\bar{u}^a\bar{u}^b} \end{array} \right. \tag{1.12}$$

between the matters variable $(z, j^a)$ and $(w, \bar{u}^a)$. We cannot give $\epsilon$, $p$, $\bar{u}^b$ and solve for $z$ and $j^\alpha$ by (1.12), since this is incompatible with the conformal scaling method (see Section 3 in [5]). Therefore we have to give $z$, $j^\alpha$ and solve for $\epsilon$ $p$, $\bar{u}^b$. Relations (1.12) are by no means trivial, and they enforce us to modify the conformal method for solving the constraint equations (see e. g. [8],

[1]). Therefore the free initial data for the Einstein-Euler system will be partially invariant under conformal transformations.

This paper deals with the solutions of the evolution equations while the construction of the initial data and the solution of the constraint equations is available as an electronic preprint in [4].

The paper is organized as follows: In the next section we perform the reduction of the Einstein-Euler system into a first order symmetric hyperbolic system. Choquet-Bruhat showed that the choice of harmonic coordinates converts the field equations (1.1) into wave equations which then can be written as a first order symmetric hyperbolic system [7], [8], [11]. Reducing the Euler equations (1.4) to a first order symmetric hyperbolic system is not a trivial matter. We use a fluid decomposition and present a new reduction of the Euler equations. Beside having a very clear geometric interpretation, we give a complete description of the structure of the characteristics conformal cone of the system, namely, it is a union of a three-dimensional hyperplane tangent to the initial manifold and the sound cone.

In Section 3 we define the weighted Sobolev spaces of fractional order $H_{s,\delta}$ and present our main results. These include a solution of the compatibility problem, the construction of initial data and a solution to the evolution equations in the $H_{s,\delta}$ spaces. The announcement of the main results has been published in [3].

The local existence for first order symmetric hyperbolic systems in $H_{s,\delta}$ is discussed in Section 4. The known existence results in the $H^s$ space [9], [15], [13], [26], [25], [18] cannot be applied to the $H_{s,\delta}$ spaces. The main difficulty here is the establishment of energy estimates for linear hyperbolic systems. In order to achieve it we have defined a specific inner-product in $H_{s,\delta}$ and in addition the Kato-Ponce commutator estimate [16], [26], [25] has an essential role in our approach. Once the energy estimates and other tools have been established in the $H_{s,\delta}$ space, we follow Majda's [18] iteration procedure and show existence, uniqueness and continuity in that norm.

Finally, in the Appendix we deal with of the construction, properties and tools for PDEs in the weighted Sobolev spaces of fractional order $H_{s,\delta}$. Triebel extended the $H_{k,\delta}$ spaces given by the norm (1.10) to a fractional order [27], [28]. We present three equivalent norms, one of which is a combination of the norm (1.10) and the norm of Lipschitz-Sobolevskij spaces [24]. This definition is essential for the understanding of the relations between the integer and the fractional order spaces (see (5.3)). However the double integral makes it almost impossible to establish any property needed for PDEs. Throughout the effort to solve this problem, we were looking for an equivalent definition of the norm: we let $\{\psi_j\}_{j=0}^{\infty}$ be a dyadic resolution of unity in $\mathbb{R}^3$ and set

$$\left(\|u\|_{H_{s,\delta}}\right)^2 = \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{2^j}\|_{H^s}^2, \tag{1.13}$$

where $(f)_\epsilon(x) = f(\epsilon x)$. When $s$ is an integer, then the norms (1.10) and (1.13) are equivalent. Our guiding philosophy is to apply the known properties of the Bessel potential spaces $H^s$ term-wise to each of the norms in the infinite sum (1.13) and in that way to extend them to the $H_{s,\delta}$ spaces. Of course, this requires a careful treatment and a sound consideration of the additional parameter $\delta$. Among the properties which we have extended to the $H_{s,\delta}$ spaces are the algebra, Moser type estimates, the embedding to the continuous and an intermediate estimate.

## 2 First Order Symmetric Hyperbolic Systems

This section deals with the reduction of the coupled evolution equations (1.1), (1.4) and (1.5) into a first order symmetric hyperbolic system.

## 2.1 The Euler equations written as a symmetric hyperbolic system

It is not obvious that the Euler equations written in the conservative form $\nabla_\alpha T^{\alpha\beta} = 0$ are symmetric hyperbolic. In fact these equations have to be transformed in order to be expressed in a symmetric hyperbolic form. Rendall presented such a transformation of these equations in [23], however, its geometrical meaning is not entirely clear and it might be difficult to generalize it to the non time symmetric case. Hence we will present a different hyperbolic reduction of the Euler equations and discuss it in some details, for we have not seen it anywhere in the literature. The basic idea is to perform the standard *fluid decomposition* and then to modify the equation by adding, in an appropriate manner, the normalization condition (1.3) which will be considered as a constraint equation.

The fluid decomposition method consists of:

1. The equation $\nabla_\nu T^{\nu\beta} = 0$ is once projected orthogonal onto $u^\alpha$ which leads to
$$u_\beta \nabla_\nu T^{\nu\beta} = 0. \tag{2.1}$$
2. The equation $\nabla_\nu T^{\nu\beta} = 0$ is projected into the rest pace $\mathcal{O}$ orthogonal to $u^\alpha$ of a fluid particle gives us:
$$P_{\alpha\beta} \nabla_\nu T^{\nu\beta} = 0 \qquad \text{with} \quad P_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta, \quad P_{\alpha\beta} u^\beta = 0. \tag{2.2}$$

Inserting this decomposition into (1.2) results in a system in the following form:

$$u^\nu \nabla_\nu \epsilon + (\epsilon + p) \nabla_\nu u^\nu = 0; \tag{2.3a}$$

$$(\epsilon + p) P_{\alpha\beta} u^\nu \nabla_\nu u^\beta + P^\nu{}_\alpha \nabla_\nu p = 0. \tag{2.3b}$$

Note that we have beside the evolution equations (2.3a) and (2.3b) the following constraint equation: $g_{\alpha\beta} u^\alpha u^\beta = -1$. We will show later, in subsection 2.1.1 that this constraint equation is conserved under the evolution equation, that is, if it holds initially at $t = t_0$, then it will hold for $t > t_0$. Note that in most textbooks, the equation (2.3b) is presented as $(\epsilon+p) g_{\alpha\beta} u^\nu \nabla_\nu u^\beta + P^\nu{}_\alpha \nabla_\nu p = 0$, which is an equivalent form, since due to the normalization condition (1.3) we have $u_\beta \nabla_\nu u^\beta = 0$.

In order to obtain a symmetric hyperbolic system we have to modify it in the following way. The normalization condition (1.3) gives that $u_\beta u^\nu \nabla_\nu u^\beta = 0$, so we add $(\epsilon + p) u_\beta u^\nu \nabla_\nu u^\beta = 0$ to equation (2.3a) and $u_\alpha u_\beta u^\nu \nabla_\nu u^\beta = 0$ to (2.3b), which together with (1.6) results in,

$$u^\nu \nabla_\nu \epsilon + (\epsilon + p) P^\nu{}_\beta \nabla_\nu u^\beta = 0 \tag{2.4a}$$

$$\Gamma_{\alpha\beta} u^\nu \nabla_\nu u^\beta + \frac{\sigma^2}{(\epsilon + p)} P^\nu{}_\alpha \nabla_\nu \epsilon = 0, \tag{2.4b}$$

where $\Gamma_{\alpha\beta} = P_{\alpha\beta} + u_\alpha u_\beta = g_{\alpha\beta} + 2u_\alpha u_\beta$. As mentioned above we will introduce a new nonlinear matter variable which is given by (1.7). The idea which is behind this is the following: The system (2.4a) and (2.4b) is almost of symmetric hyperbolic form, it would be symmetric if we multiply the system by appropriate factors, for example, (2.4a) by $\frac{\partial p}{\partial \epsilon} = \sigma^2$ and (2.4b) by $(\epsilon + p)$. However, doing so we will be faced with a system in which the coefficients will either tend to zero or to infinity, as $\epsilon \to 0$. Hence, it is impossible to represent this system in a non-degenerate form using these multiplications.

The central point is now to introduce a new variable $w = M(\epsilon)$ which will regularize the equations even for $\epsilon = 0$. We do this by multiplying equation (2.4a) by $\kappa^2 M' = \kappa^2 \frac{\partial M}{\partial \epsilon}$. This results in the following system which we have written in matrix form:

$$\left( \begin{array}{c|c} \kappa^2 u^\nu & \kappa^2 (\epsilon + p) M' P^\nu{}_\beta \\ \hline \frac{\sigma^2}{(\epsilon+p)M'} P^\nu{}_\alpha & \Gamma_{\alpha\beta} u^\nu \end{array} \right) \nabla_\nu \begin{pmatrix} w \\ u^\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.5}$$

In order to obtain symmetry we have to demand

$$M' = \frac{\sigma}{(\epsilon + p)\kappa}, \tag{2.6}$$

where $\kappa \gg 0$ has been introduced in order to simplify the expression for $w$. We choose $\kappa$ so that

$$\frac{\sqrt{f'(\epsilon)}}{(\epsilon + p)\kappa} = \frac{2}{\gamma - 1}\frac{\epsilon^{\frac{\gamma-1}{2}}}{\epsilon}, \tag{2.7}$$

which gives the Makino variable (1.7). Taking into account the equation of state (1.5), we see that

$$\kappa = \frac{\gamma - 1}{2}\frac{\sqrt{K\gamma}}{1 + K\epsilon^{\gamma-1}} \gg 0. \tag{2.8}$$

Finally we have obtained the following system

$$\left(\begin{array}{c|c} \kappa^2 u^\nu & \sigma\kappa {P^\nu}_\beta \\ \hline \kappa\sigma {P^\nu}_\alpha & \Gamma_{\alpha\beta}u^\nu \end{array}\right) \nabla_\nu \begin{pmatrix} w \\ u^\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.9}$$

which is both symmetric and non-degenerated. The covariant derivative $\nabla_\nu$ takes in local coordinates the form $\nabla_\nu = \partial_\nu + \Gamma(g^{\gamma\delta}, \partial g_{\alpha\beta})$ which expresses the fact that the fluid $u^\alpha$ is coupled to equations (1.1) for the gravitational field $g_{\alpha\beta}$. In addition, from the definition of the Makino variable (1.7) we see that $\epsilon^{\gamma-1} = w^2$, so from the expression (1.6), is follows that $\sigma = \sqrt{\gamma K}w$ and $\kappa$ –which is given by (2.8)– are $C^\infty$ functions of $w$. Thus the fractional power of the equation of state (1.5) does not appear in the coefficients of the system (2.9), and these coefficients are $C^\infty$ functions of the scalar $w$, the four vector $u^\alpha$ and the gravitational field $g_{\alpha\beta}$.

Let us now recall a general definition of symmetric hyperbolic systems.

**Definition 2.1** ***(First order symmetric hyperbolic systems)*** *A quasilinear, symmetric hyperbolic system is a system of differential equations of the form*

$$L[U] = \sum_{\alpha=0}^{4} A^\alpha(U;x)\partial_\alpha U + B(U;x) = 0 \tag{2.10}$$

*where the matrices $A^\alpha$ are symmetric and for every arbitrary $U \in G$ there exists a covector $\xi$ such that*

$$\xi_\alpha A^\alpha(U;x) \tag{2.11}$$

*is positive definite. The covectors $\xi_\alpha$ for which (2.11) is positive definite, are spacelike with respect to the equation (2.10). Both matrices $A^\alpha$, $B$ satisfy certain regularity conditions, which are going to be formulated later.*

Usually $\xi$ is chosen to be the vector $(1,0,0,0)$ which implies via the condition (2.11) that the matrix $A^0$ has to be is positive definite.

Now we want to show that $A^0$ of our system (2.9) is indeed positive definite. We do this in several steps.

1. Explicit computation of the principle symbol (2.9);
2. We show that $-u_\alpha$ is a space like covector with respect to the equations;
3. Then we apply a deformation argument and show that the covector $t_\alpha := (1,0,0,0)$ is a space like covector with respect to the equation.

For each $\xi_\alpha \in T_x^*V$ the principle symbol is a linear map from $\mathbb{R}\times E_x$ to $\mathbb{R}\times F_x$, where $E_x$ is a fiber in $T_xV$ and $F_x$ is a fiber in the cotangent space $T_x^*V$. Since in local coordinates $\nabla_\nu = \partial_\nu + \Gamma(g^{\gamma\delta}, \partial g_{\alpha\beta})$, the principle symbol of system (2.9) is

$$\xi_\nu A^\nu = \left(\begin{array}{c|c} \kappa^2(u^\nu\xi_\nu) & \sigma\kappa P^\nu{}_\beta\xi_\nu \\ \hline \sigma\kappa P^\nu{}_\alpha\xi_\nu & (u^\nu\xi_\nu)\Gamma_{\alpha\beta} \end{array}\right) \tag{2.12}$$

and the characteristics are the set of covectors for which $(\xi_\nu A^\nu)$ is not an isomorphism. Hence the characteristics are the zeros of $Q(\xi) := \det(\xi_\nu A^\nu)$.

The geometric advantages of the fluid decomposition are the following. The operators in the blocks of the matrix (2.12) are $P^\nu{}_\alpha$, the projection on the rest hyperplane $\mathcal{O}$ and $\Gamma_{\alpha\beta}$, that is the reflection with respect to the same hyperplane. Therefore, the following relations hold:

$$\Gamma^{\alpha\gamma}\Gamma_{\gamma\beta} = \delta_\beta{}^\alpha, \qquad \Gamma^{\alpha\gamma}P_\gamma{}^\nu = P^{\alpha\nu} \qquad \text{and} \qquad P_\beta{}^\alpha P_\alpha{}^\nu = P^\nu{}_\beta,$$

which yields

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \Gamma^{\alpha\gamma} \end{array}\right)(\xi_\nu A^\nu) = \left(\begin{array}{c|c} \kappa^2(u^\nu\xi_\nu) & \sigma\kappa P^\nu{}_\beta\xi_\nu \\ \hline \sigma\kappa P^{\alpha\nu}\xi_\nu & (u^\nu\xi_\nu)\left(\delta^\alpha_\beta\right) \end{array}\right). \tag{2.13}$$

It is now fairly easy to calculate the determinate of the right hand side of (2.13) and we have

$$\det\left(\begin{array}{c|c} \kappa^2(u^\nu\xi_\nu) & \sigma\kappa P^\nu{}_\beta\xi_\nu \\ \hline \sigma\kappa P^{\alpha\nu}\xi_\nu & (u^\nu\xi_\nu)\left(\delta^\alpha_\beta\right) \end{array}\right) = \kappa^2(u^\nu\xi_\nu)^3\left((u^\nu\xi_\nu)^2 - \sigma^2 P^{\alpha\nu}\xi_\nu P^\nu_\alpha\xi_\nu\right).$$

Since $P^\alpha_\beta$ is a projection,

$$P^{\alpha\nu}\xi_\nu P^\nu_\alpha\xi_\nu = g^{\nu\beta}\xi_\nu P^\alpha_\beta P^\nu_\alpha\xi_\nu = g^{\nu\beta}\xi_\nu P^\nu{}_\beta\xi_\nu = P^\nu{}_\beta\xi_\nu\xi^\beta \tag{2.14}$$

and since $\Gamma^\gamma_\beta : \mathbb{R}^4 \to \mathbb{R}^4$ is a reflection with respect to a hyperplane,

$$\det\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \Gamma^{\alpha\gamma} \end{array}\right) = \det\left(g^{\alpha\beta}\Gamma^\gamma_\beta\right) = \det\left(g^{\alpha\beta}\right)\det\left(\Gamma^\gamma_\beta\right) = -\left(\det\left(g_{\alpha\beta}\right)\right)^{-1}. \tag{2.15}$$

Consequently,

$$Q(\xi) := \det(\xi_\nu A^\nu) = -\kappa^2\det(g_{\alpha\beta})(u^\nu\xi_\nu)^3\left\{(u^\nu\xi_\nu)^2 - \sigma^2 P^\alpha{}_\beta\xi_\alpha\xi^\beta\right\} \tag{2.16}$$

and therefore the characteristic covectors are given by two simple equations:

$$\xi_\nu u^\nu = 0; \tag{2.17}$$
$$(\xi_\nu u^\nu)^2 - \sigma^2 P^\alpha{}_\beta\xi_\alpha\xi^\beta = 0. \tag{2.18}$$

**Remark 2.2** ***(The structure of the characteristics conormal cone of )*** *The characteristics conormal cone is therefore a union of two hypersurfaces in $T_x^*V$. One of these hypersurfaces is given by the condition (2.17) and it is a three dimensional hyperplane $\mathcal{O}$ with the normal $u^\alpha$. The other hypersurface is given by the condition (2.18) and forms a three dimensional cone the so called sound cone.*

**Remark 2.3** *Equation (2.18) plays an essential role in determining whether the equations form a symmetric hyperbolic system.*

Let us now consider the timelike vector $u_\nu$ and the linear combination $-u_\nu A^\nu$, with $A^\nu$ from equation (2.9), we then obtain that

$$-u_\nu A^\nu = \left(\begin{array}{c|c} \kappa^2 & 0 \\ \hline 0 & \Gamma_{\alpha\beta} \end{array}\right) \tag{2.19}$$

is positive definite. Indeed, $\Gamma_{\alpha\beta}$ is a reflection with respect to a hyperplane. The normal of this hyperplane is a timelike vector. Hence, $-u_\nu$ is for the hydrodynamical equations a spacelike covector in the sense of partial differential equations. Herewith one has showed relatively elegant and elementary that the relativistic hydrodynamical equations are symmetric–hyperbolic.

Now we want however to show that the covector $t_\alpha = (1,0,0,0)$ is spacelike with respect to the system (2.9). Since $P^\alpha{}_\beta u_\alpha = 0$, the covector $-u_\nu$ belongs to the sound cone

$$(\xi_\nu u^\nu)^2 - \sigma^2 P^\alpha{}_\beta \xi_\alpha \xi^\beta > 0. \tag{2.20}$$

Inserting $t_\nu = (1,0,0,0)$ the right hand side of (2.20) yields

$$(u^0)^2(1-\sigma^2) - \sigma^2 g^{00}. \tag{2.21}$$

Since the sound velocity is always less than the light speed, that is $\sigma^2 = \frac{\partial p}{\partial \epsilon} < c^2 = 1$, we conclude from (2.21) that $t_\nu$ also belongs to the sound cone (2.20). Hence, the vector $-u_\nu$ can be continuously deformed to $t_\nu$ while condition (2.20) holds along the deformation path. Consequently, the determinant of (2.16) remains positive under this process and hence $t_\nu A^\nu = A^0$ is also positive definite.

### 2.1.1 Conservation of the constraint equation $g_{\alpha\beta}u^\alpha u^\beta = -1$

Now it will be shown that the condition $g_{\alpha\beta}u^\alpha u^\beta = -1$, which acts as a constraint equation for the evolution equation, is conserved along stream lines $u^\alpha$. Because, if for $t = t_0$ the condition $g_{\alpha\beta}u^\alpha u^\beta = -1$ holds and if it is conserved a long stream lines, then $g_{\alpha\beta}u^\alpha u^\beta = -1$ holds also for $t > t_0$. So let $c(t)$ be a curve such that $c'(t) = u^\alpha$ and set $Z(t) = (u \circ c)_\beta (u \circ c)^\beta$, then we need to establish

$$\frac{d}{dt} Z(t) = 2u_\beta \nabla_{c'(t)} u^\beta = 2u^\nu u_\beta \nabla_\nu u^\beta = 0. \tag{2.22}$$

Multiplying the last four last rows of the Euler system (2.9) by $u^\alpha$ and recalling that $P^\nu{}_\alpha$ is the projection on the rest space $\mathcal{O}$ orthogonal to $u^\alpha$, we have

$$\begin{aligned} 0 &= u^\alpha \left(\Gamma_{\alpha\beta} u^\nu \nabla_\nu u^\beta + \kappa\sigma P^\nu{}_\alpha \nabla_\nu w\right) \\ &= u^\alpha P_{\alpha\beta} u^\nu \nabla_\nu u^\beta - u^\nu u_\beta \nabla_\nu u^\beta + \kappa\sigma u^\alpha P^\nu{}_\alpha \nabla_\nu w \\ &= -u^\nu u_\beta \nabla_\nu u^\beta. \end{aligned}$$

## 2.2 The reduced Einstein field equations

In this paper we study the field equations (1.1) with the choice of the harmonic coordinate condition which takes the form

$$H^\alpha \equiv g^{\alpha\beta} g^{\gamma\delta} (\partial_\gamma g_{\beta\delta} - \frac{1}{2} \partial_\delta g_{\beta\gamma}) = 0. \tag{2.23}$$

When (2.23) is imposed, then the Einstein equations (1.1) convert to

$$g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} = H_{\alpha\beta}(g,\partial g) - 16\pi T_{\alpha\beta} + 8\pi g^{\mu\nu}T_{\mu\nu}g_{\alpha\beta}. \tag{2.24}$$

Hawking and Ellis proved the conservation of the harmonic coordinates for Einstein equations with matter including a perfect fluid [11]. Since (2.24) are quasi linear wave equations, the introducing auxiliary variables

$$h_{\alpha\beta\gamma} = \partial_\gamma g_{\alpha\beta}, \tag{2.25}$$

reduce them into a first order symmetric hyperbolic system:

$$\begin{array}{rcl} \partial_t g_{\alpha\beta} & = & h_{\alpha\beta 0} \\ g^{ab}\partial_t h_{\gamma\delta a} & = & g^{ab}\partial_a h_{\gamma\delta 0} \\ -g^{00}\partial_t h_{\gamma\delta 0} & = & 2g^{0a}\partial_a h_{\gamma\delta 0} + g^{ab}\partial_a h_{\gamma\delta b} \\ & & +C^{\epsilon\zeta\eta\kappa\lambda\mu}_{\gamma\delta\alpha\beta\rho\sigma} h_{\epsilon\zeta\eta}h_{\kappa\lambda\mu}g^{\alpha\beta}g^{\rho\sigma} - 16\pi T_{\gamma\delta} + 8\pi g^{\rho\sigma}T_{\rho\sigma}g_{\gamma\delta} \end{array} \tag{2.26}$$

The object $C^{\epsilon\zeta\eta\kappa\lambda\mu}_{\gamma\delta\alpha\beta\rho\sigma}$ is a combination of Kronecker deltas with integer coefficients. We therefore conclude:

**Conclusion 2.4** ***(The evolution equations in a first order symmetric hyperbolic form)*** *The equations for Einstein gravitational field (1.1) coupled with the Euler equations (1.4) with the normalization conditions (1.3) and the equation of state (1.5), are equivalent to the system (2.26) and (2.9). The coupled systems (2.26) and (2.9) take the form of a first order symmetric hyperbolic system in accordance with Definition 2.1 and where $A^0$ is a positive definite matrix.*

# 3 New Function Spaces and the Principle Results

Our principle results concern the solution to the coupled evolution equations (1.1) and (1.4), for which we have shown that they are equivalent to the first order symmetric hyperbolic systems (2.9) and (2.26). The initial data for these coupled systems cannot be given freely, therefore they are constructed in the following way. Firstly the compatibility of the initial data for the fluid and the gravitational field (1.12) have to be solved and next the constraint equations (1.11), which lead to an elliptic system. For the convenience of the reader we include here the construction of the initial data. The proofs of Theorems 3.2 and 3.4 below appear in [5] and they are also available as an electronic preprint in [4].

We first define the *weighted fractional Sobolev spaces.* We make a dyadic resolution of the unity in $\mathbb{R}^3$ as follows. Let $K_j = \{x : 2^{j-3} \le |x| \le 2^{j+2}\}$, $(j = 1, 2, ...)$ and $K_0 = \{x : |x| \le 4\}$. Let $\{\psi_j\}_{j=0}^{\infty}$ be a sequence of $C_0^\infty(\mathbb{R}^3)$ such that $\psi_j(x) = 1$ on $K_j$, $\mathrm{supp}(\psi_j) \subset \cup_{l=j-4}^{j+3} K_l$, for $j \ge 1$ and $\mathrm{supp}(\psi_0) \subset K_0 \cup K_1$.

We denote by $H^s$ the Bessel potential spaces with the norm $(p = 2)$

$$\|u\|^2_{H^s} = c\int (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi,$$

where $\hat{u}$ is the Fourier transform of $u$. Also, for a function $f$, $f_\varepsilon(x) = f(\varepsilon x)$.

**Definition 3.1** ***(Weighted fractional Sobolev spaces: infinite sum of semi norms)*** *For $s \ge 0$ and $-\infty < \delta < \infty$,*

$$\left(\|u\|_{H_{s,\delta}}\right)^2 = \sum_j 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{(2^j)}\|^2_{H^s}. \tag{3.1}$$

*The space $H_{s,\delta}$ is the set of all temperate distributions with a finite norm given by (3.1).*

## 3.1 The principle results

### 3.1.1 The compatibility of the initial data for the fluid and the gravitational field

The matter data (non-gravitational) $(z, j)$ which appear in the right hand side of (1.11) are coupled to the initial data of the perfect fluid (1.2) via the relations (1.12). Thus, an indispensable condition for obtaining a solution of the Einstein-Euler system is the inversion of (1.12). This system is not invertible for all $(z, j^a) \in \mathbb{R}_+ \times \mathbb{R}^3$, but the inverse does exist in a certain region.

**Theorem 3.2** ***(Reconstruction theorem for the initial data)*** *There is a real function $S : [0, 1) \to \mathbb{R}$ such that if*

$$0 \leq z < S(\sqrt{h_{ab}j^a j^b}/z), \tag{3.2}$$

*then system (1.12) has a unique inverse. Moreover, the inverse mapping is continuous in $H_{s,\delta}$ norm.*

**Remark 3.3** *The matter initial data $(z, j^a)$ for the Einstein-Euler system with the equation of state (1.5) cannot be given freely. They must satisfy condition (3.2). This condition includes the inequality*

$$z^2 \geq h_{ab}j^a j^b, \tag{3.3}$$

*which is known as the dominate energy condition.*

### 3.1.2 Solution to the constraint equations

The gravitational data is a triple $(M, h, K)$, where $M$ is a space-like asymptotically flat manifold, $h = h_{ab}$ is a proper Riemannian metric on $M$, and $K = K_{ab}$ is the second fundamental form on $M$ (extrinsic curvature). The metric $h_{ab}$ and the extrinsic curvature $K$ must satisfy Einstein's constraint equations (1.11). The free initial data is a set $(\bar{h}_{ab}, \bar{A}_{ab}, \hat{y}, \hat{v}^a)$, where $\bar{h}_{ab}$ is a Riemannian metric, $\bar{A}_{ab}$ is divergence and trace free form, $\hat{y}$ is a scalar function and $\hat{v}^a$ is a vector.

**Theorem 3.4** ***(Solution of the constraint equations)*** *Given free data $(\bar{h}_{ab}, \bar{A}_{ab}, \hat{y}, \hat{v}^a)$ such that $(\bar{h}_{ab} - I) \in H_{s,\delta}$, $\bar{A}_{ab} \in H_{s-1,\delta+1}$, $(\hat{y}, \hat{v}^a) \in H_{s-1,\delta+2}$, $\frac{5}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$.*

*(i) Then there exists two positive functions $\alpha$ and $\phi$ such that $(\alpha - 1), (\phi - 1) \in H_{s,\delta}$, a vector field $W \in H_{s,\delta}$ such that the gravitational data*

$$h_{ab} = (\phi\alpha)^4 \bar{h}_{ab} \qquad \text{and} \qquad K_{ab} = (\phi\alpha)^{-2}\bar{A}_{ab} + \phi^{-2}\hat{\mathcal{L}}(W) \tag{3.4}$$

*satisfy the constraint equations (1.11) with $z = \phi^{-8}\hat{y}^{\frac{2}{\gamma-1}}$ and $j^b = \phi^{-10}\hat{y}^{\frac{2}{\gamma-1}}\hat{v}^b$ as the right hand side, here $\hat{\mathcal{L}}$ is the Killing vector field operator. In addition, the $H_{s,\delta} \times H_{s-1,\delta+1}$ norms of $(h_{ab} - I, K_{ab})$ depend continuously on the $H_{s,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$ norms of $(\bar{h}_{ab} - I, \bar{A}_{ab}, \hat{y}, \hat{j}^a)$.*

*(ii) Let $\hat{h}_{ab} = \alpha^4 \bar{h}_{ab}$, $\hat{z} = \hat{y}^{\frac{2}{\gamma-1}}$, $j^a = \hat{y}^{\frac{2}{\gamma-1}}\hat{v}^a$ and $\Omega^{-1}$ denote the inverse of relations (1.12). If $(\hat{h}_{ab}, \hat{z}, \hat{j}^a)$ satisfies (3.2), then the data for the four velocity vector and Makino variable are given by: $z = \phi^{-8}\hat{z}$, $j^a = \phi^{-10}\hat{j}^a$,*

$$(w, \bar{u}^a) := \Omega^{-1}(z, j^a) \qquad \text{and} \qquad \bar{u}^0 = 1 + h_{ab}\bar{u}^a\bar{u}^b \tag{3.5}$$

*and they satisfy the compatibility conditions (1.12). In addition, the $H_{s-1,\delta+2}$ norms of $(w, \bar{u}^a, u^0 - 1)$ depend continuously on the $H_{s,\delta} \times H_{s-1,\delta+2}$ norms of $(\bar{h}_{ab} - I, \hat{y}, \hat{j}^a)$.*

#### 3.1.3 Solution to the evolution equations

The unknowns of the evolution equations are the gravitational field $g_{\alpha\beta}$ and its first order partial derivatives $\partial_\alpha g_{\gamma\delta}$, the Makino variable $w$ and the velocity vector $u^\alpha$. We represent them by the vector $U = \left(g_{\alpha\beta} - \eta_{\alpha\beta}, \partial_a g_{\gamma\delta}, \partial_0 g_{\gamma\delta}, w, u^a, u^0 - 1\right)$, here $\eta_{\alpha\beta}$ denotes the Minkowski metric. The initial data for equation (2.26) are

$$g_{ab|_M} = h_{ab}, \quad g_{0b|_M} = 0, \quad g_{00|_M} = -1, \quad -\frac{1}{2}\partial_0 g_{ab|_M} = K_{ab}, \tag{3.6}$$

where $(h_{ab}, K_{ab})$ are given by (3.4), and (3.5) for equation (2.9). Theorem 3.4 guarantees that they satisfy the constraints (1.11) and the compatibility condition (1.12).

**Theorem 3.5** ***(Solutions of the evolution equations (2.26) and (2.9))*** *Let $\frac{7}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$. Given the solutions of the constraint equations as described in Theorem 3.4, then there exists a $T > 0$, a unique semi-Riemannian metric $g_{\alpha\beta}$ solution to (2.26) and a unique pair $(w, u^\alpha)$ solution to (2.9) such that*

$$(g_{\alpha\beta} - \eta_{\alpha\beta}) \in C\left([0,T], H_{s,\delta}\right) \cap C^1\left([0,T], H_{s-1,\delta+1}\right) \tag{3.7}$$

$$(w, u^a, u^0 - 1) \in C\left([0,T], H_{s-1,\delta+2}\right) \cap C^1\left([0,T], H_{s-2,\delta+3}\right). \tag{3.8}$$

# 4 Local Existence for Hyperbolic Equations

In this section we prove an existence theorem (locally in time) for quasi linear symmetric hyperbolic system in the $H_{s,\delta}$ spaces. The known existence results in the $H^s$ space of Fisher and Marsden [9] and Kato [15] (see also [26], [25]), cannot be applied to the $H_{s,\delta}$ spaces. The main difficulty here is the establishment of energy estimates for linear hyperbolic systems. In order to achieve it we have defined a specific inner-product in $H_{s,\delta}$ (see Definition 4.3) and in addition the Kato-Ponce commutator estimate [16], [26], has an essential role in our approach. Once the energy estimates have been established in the $H_{s,\delta}$ space, we follow Majda's [18] iteration procedure and show existence, uniqueness and continuity in that norm.

We consider the Cauchy problem for a quasi linear (uniform) symmetric hyperbolic system of the form

$$\begin{cases} A^0(u;t,x)\partial_t u + \sum_{a=1}^{3} A^a(u;t,x)\partial_a u + B(u;t,x)u + F(u;t,x) = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{4.1}$$

under the following assumptions:

(H1) $A^\alpha$ are symmetric matrices for $\alpha = 0,1,2,3$;

(H2) $A^\alpha(u;t,x), B(u;t,x), F(u;t,x)$ are smooth in their arguments;

(H3) $\left(A^0(0;t,\cdot) - I\right), A^a(0;t,\cdot), B(0;t,\cdot), F(0;t,\cdot) \in H_{s,\delta}$;

(H4) $\partial_t A^0(u;t,\cdot) \in L^\infty$.

The main result of this section is the well posedness of the system (4.1) in $H_{s,\delta}$ spaces:

**Theorem 4.1** ***(Well posedness of first order hyperbolic symmetric systems in $H_{s,\delta}$)*** *Let $s > \frac{5}{2}$, $\delta \geq -\frac{3}{2}$ and assume hypotheses (H1)-(H4) holds. If the initial condition $u_0$ belongs to $H_{s,\delta}$ and satisfies*

$$\frac{1}{\mu}\delta_{\alpha\beta}u_0^\alpha u_0^\beta \leq A^0_{\alpha\beta}u_0^\alpha u_0^\beta \leq \mu\delta_{\alpha\beta}u_0^\alpha u_0^\beta, \qquad \mu \in \mathbb{R}^+ \tag{4.2}$$

*then there exits a positive $T$ which depends on the $H_{s,\delta}$-norm of the initial data and there exists a unique $u(t,x)$ a solution to (4.1) which in addition satisfies*

$$u \in C([0,T], H_{s,\delta}) \cap C^1([0,T], H_{s-1,\delta+1}). \tag{4.3}$$

**Remark 4.2** *Condition (H3) is sometime too restrictive for applications. We may replace it by*

*(H3')* $\left(A^0(U^0;t,\cdot) - I\right), A^a(U^0;t,\cdot), B(U^0;t,\cdot), F(U^0;t,\cdot) \in H_{s,\delta}$,

*where $U^0$ is a constant vector. Setting $u = U^0 + v$, then $v$ satisfies*

$$\begin{cases} \tilde{A}^0(v;t,x)\partial_t v = \sum_{a=1}^3 \tilde{A}^a(u;t,x)\partial_a v + \tilde{B}(v;t,x)v + \tilde{F}(v;t,x) \\ v(0,x) = u_0(x) - U^0 \end{cases} \tag{4.4}$$

*where $\tilde{A}^\alpha(v;t,x) = A^\alpha(U^0+v;t,x)$, $\tilde{B}(v;t,x) = B(U^0+v;t,x)$ and $\tilde{F}(v;t,x) = F(U^0+v;t,x) + \tilde{B}(U^0+v;t,x)$. The Moser type estimates are valid under assumptions (H3') (see Remark 6.7).*

## 4.1 Strategy

We will proceed with the following strategy:

1. The establishment of energy estimates for linear systems in the fractional weighted spaces $H_{s,\delta}$.
2. We approximate the initial data by a $C_0^\infty$ sequence and then construct an iteration process which consists of solutions to a linear system having a $C_0^\infty$ initial data.
3. We show that the sequence which is constructed by the iteration process is bounded in $H_{s,\delta}$-norm and weakly converges to a solution.
4. At the final stage we prove uniqueness and continuity in $H_{s,\delta}$-norm.

## 4.2 Energy estimates in the fractional weighted spaces

The energy estimates are indispensable means for the proof of well posedness of hyperbolic systems. In order to achieve it we introduce an inner product which depends on a matrix $A$. We assume $A = A(t,x)$ is $m \times m$ symmetric matrix which satisfies

$$\frac{1}{\mu} U^T U \le U^T A U \le \mu U^T U \tag{4.5}$$

for some positive $\mu$. Here $B^T$ denotes the transpose matrix. We recall that $f_\epsilon(x) = f(\epsilon x)$, the sequence $\{\psi_j\}$ is a dyadic resolution of the unity in $\mathbb{R}^3$ which is defined the Appendix 5 and that $\Lambda^s u = \mathcal{F}^{-1}\left((1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}u\right)$, where $\mathcal{F}$ denotes the Fourier transform. In this section the expression (4.6) below will serve as a norm of the space $H_{s,\delta}$:

$$\|u\|^2_{H_{s,\delta}} := \sum_{j=0}^\infty 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j^2 u)_{(2^j)}\|^2_{H^s}. \tag{4.6}$$

Corollary 5.6 implies that (4.6) is equivalent to the norm of Definition 3.1.

**Definition 4.3** ***(Inner Product)*** *For a symmetric matrix $A = A(t,x)$ which satisfies (4.5) we let*

$$\begin{aligned}\langle u,v\rangle_{s,\delta,A} &:= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\left\langle \Lambda^s\left((\psi_j^2 u)_{(2^j)}\right),(A)_{2^j}\,\Lambda^s\left((\psi_j^2 v)_{(2^j)}\right)\right\rangle_{L^2}\\ &= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\int\left[\Lambda^s\left((\psi_j^2 u)_{(2^j)}\right)\right]^T(A)_{2^j}\left[\Lambda^s\left((\psi_j^2 v)_{(2^j)}\right)\right]dx \qquad (4.7)\end{aligned}$$

*and its associated norm $\|u\|^2_{H_{s,\delta,A}} = \langle u,u\rangle_{s,\delta,A}$.*

Obviously $\langle u,v\rangle_{s,\delta,A} = \langle v,u\rangle_{s,\delta,A}$ and from (4.5) we obtain the equivalence,

$$\frac{1}{\mu}\|u\|^2_{H_{s,\delta}} \le \|u\|^2_{H_{s,\delta,A}} \le \mu\|u\|^2_{H_{s,\delta}}. \qquad (4.8)$$

We come now to the crucial estimate of this section.

**Lemma 4.4** ***(An energy estimate)*** *Let $s > \frac{5}{2}$, $\delta \ge -\frac{3}{2}$, $A^\alpha = A^\alpha(t,x)$ be $m\times m$ symmetric matrices such that $(A^0(t,\cdot)-I), A^a(t,\cdot) \in H_{s,\delta}$ and $A^0$ satisfies (4.5). If $u(t) = u(t,\cdot)$ is a $C_0^\infty$ solution of the linear hyperbolic system*

$$A^0(t,x)\partial_t u = \sum_{a=1}^{3} A^a(t,x)\partial_a u, \qquad (4.9)$$

*then*

$$\frac{d}{dt}\|u(t)\|^2_{H_{s,\delta,A^0}} \le C\left(\mu\|u(t)\|^2_{H_{s,\delta,A^0}} + 1\right), \qquad (4.10)$$

*where $C = C(\left\|A^0 - I\right\|_{H_{s,\delta}}, \|A^a\|_{H_{s,\delta}}, \|\partial_t u\|_{H_{s-1,\delta}}, \|\partial_t A^0\|_{L^\infty})$.*

An essential tool for deriving these estimates is the Kato & Ponce Commutator Estimate [16], [26].

**Theorem 4.5** ***(Kato and Ponce)*** *Let $s > 0$, $f \in H^s \cap C^1$, $g \in H^{s-1}\cap L^\infty$, then*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^2} \le C\left\{\|\nabla f\|_{L^\infty}\|g\|_{H^{s-1}} + \|f\|_{H^s}\|g\|_{L^\infty}\right\}. \qquad (4.11)$$

This estimate will be used term wise in the inner product (4.7).

**Proof (of Lemma 4.4)** Since $u$ is $C_0^\infty$ we may interchange the derivation with respect to $t$ with the inner-product (4.7) and get

$$\begin{aligned}\frac{d}{dt}\langle u,u\rangle_{s,\delta,A^0} &= 2\langle u,\partial_t u\rangle_{s,\delta,A^0}\\ &+ \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\int\left[\Lambda^s\left((\psi_j^2 u)_{(2^j)}\right)\right]^T\left(\partial_t A^0\right)_{2^j}\left[\Lambda^s\left((\psi_j^2 u)_{(2^j)}\right)\right]dx\\ &\le 2\langle u,\partial_t u\rangle_{s,\delta,A^0} + \|\partial_t A^0\|_{L^\infty}\left(\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\|(\psi_j^2 u)_{(2^j)}\|^2_{H^s}\right)\\ &= 2\langle u,\partial_t u\rangle_{s,\delta,A^0} + \|\partial_t A^0\|_{L^\infty}\|u\|^2_{H_{s,\delta}} \qquad (4.12)\end{aligned}$$

We turn now to the hard task of the proof, namely, the estimation of $\langle u,\partial_t u\rangle_{s,\delta,A^0}$. Put

$$E(j) = \left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \left((A^0)_{2^j}\right)\Lambda^s\left(\left(\psi_j^2\partial_t u\right)_{2^j}\right)\right\rangle_{L^2} \qquad (4.13)$$

and let $\{\Psi_k\} = \frac{1}{\sum \psi_j(x)} \psi_k(x)$, where $\{\psi_j\}$ is defined in Appendix 5. It follows from the properties of this sequence that

$$\Psi_k \psi_j^2 \neq 0 \qquad \text{only when } k = j-3, ..., j+4. \tag{4.14}$$

Hence,

$$\begin{aligned}
E(j) &= \left\langle \Lambda^s \left( \left(\psi_j^2 u\right)_{2^j} \right), \left((A^0)_{2^j}\right) \Lambda^s \left( \left( \sum_{k=0}^{\infty} \Psi_k \right)_{2^j} \left(\psi_j^2 \partial_t u\right)_{2^j} \right) \right\rangle_{L^2} \\
&= \sum_{k=j-3}^{j+4} \left\langle \Lambda^s \left( \left(\psi_j^2 u\right)_{2^j} \right), \left((A^0)_{2^j}\right) \Lambda^s \left( (\Psi_k)_{2^j} \left(\psi_j^2 \partial_t u\right)_{2^j} \right) \right\rangle_{L^2} \\
&= \sum_{k=j-3}^{j+4} \left\langle \Lambda^s \left( \left(\psi_j^2 u\right)_{2^j} \right), (A^0)_{2^j} \left[ \Lambda^s \left( (\Psi_k)_{2^j} \left(\psi_j^2 \partial_t u\right)_{2^j} \right) - (\Psi_k)_{2^j} \Lambda^s \left(\psi_j^2 \partial_t u\right)_{2^j} \right] \right\rangle_{L^2} \\
&+ \sum_{k=j-3}^{j+4} \left\langle \Lambda^s \left( \left(\psi_j^2 u\right)_{2^j} \right), (\Psi_k A^0)_{2^j} \Lambda^s \left(\psi_j^2 \partial_t u\right)_{2^j} \right\rangle_{L^2} \\
&= E_1(j,k) + E_2(j,k).
\end{aligned}$$

This splitting will enable us to estimate $E_2(j,k)$ in terms of the $H_{s,\delta}$ norm of $A^0 - I$ while by Theorem 4.5,

$$\begin{aligned}
& |E_1(j,k)| \\
\leq \; & \left\| \Lambda^s \left( \left(\psi_j^2 u\right)_{2^j} \right) \right\|_{L^2} \|A^0\|_{L^\infty} \left\| \Lambda^s \left( (\Psi_k)_{2^j} \left(\psi_j^2 \partial_t u\right)_{2^j} \right) - (\Psi_k)_{2^j} \Lambda^s \left(\psi_j^2 \partial_t u\right)_{2^j} \right\|_{L^2} \\
\leq \; & \left\| \left(\psi_j^2 u\right)_{2^j} \right\|_{H^s} \|A^0\|_{L^\infty} \left\{ \|\nabla (\Psi_k)_{2^j}\|_{L^\infty} \left\| \left(\psi_j^2 \partial_t u\right)_{2^j} \right\|_{H^{s-1}} + \|(\Psi_k)_{2^j}\|_{H^s} \left\| \left(\psi_j^2 \partial_t u\right)_{2^j} \right\|_{L^\infty} \right\} \\
\leq \; & C \|A^0\|_{L^\infty} \left( \left\| \left(\psi_j^2 u\right)_{2^j} \right\|_{H^s} \left\| \left(\psi_j^2 \partial_t u\right)_{2^j} \right\|_{H^{s-1}} \right).
\end{aligned} \tag{4.15}$$

In the last step above we have used the below useful estimates. First, by (5.4) and (4.14),

$$\|\nabla (\Psi_k)_{2^j}\|_{L^\infty} = 2^j \|\nabla \Psi_k\|_{L^\infty} \leq C 2^j 2^{-k} \leq 8C. \tag{4.16}$$

Secondly, it is well known that for any smooth function $f$,

$$\|fu\|_{H^s} \leq C(\|f\|_{C^N}) \|u\|_{H^s}, \tag{4.17}$$

where the integer $N$ is not less than $s$. In addition, from (5.12) we see that

$$\|f_\epsilon\|_{H^s}^2 \lesssim \begin{cases} \epsilon^{-3} \|f\|_{H^s}^2, & \epsilon \leq 1 \\ \epsilon^{2s-3} \|f\|_{H^s}^2, & \epsilon \geq 1 \end{cases}. \tag{4.18}$$

Recalling that $\psi_k(x) = \psi_1(2^{-k}x)$ and $(\psi_k(x))_{2^j} = (\psi_1(x))_{2^{j-k}}$, applying the above and combing this with (4.14) and (4.17), we have

$$\begin{aligned}
\|(\Psi_k)_{2^j}\|_{H^s} &= \left\| \left( \sum_j \psi_j \right)_{2^j}^{-1} (\psi_k)_{2^j} \right\|_{H^s} \leq C \|(\psi_k)_{2^j}\|_{H^s} \\
&= C \left\| (\psi_1)_{2^(j-k)} \right\|_{H^s} \leq C 2^{(s-\frac{3}{2})3} \|\psi_1\|_{H^s}.
\end{aligned} \tag{4.19}$$

Finally, by the Sobolev embedding

$$\|v\|_{L^\infty} \leq C \|v\|_{H^s}, \tag{4.20}$$

we obtain $\left\|\left(\psi_j^2\partial_t u\right)_{2^j}\right\|_{L^\infty} \le C\left\|\left(\psi_j^2\partial_t u\right)_{2^j}\right\|_{H^{s-1}}$.

In order to use equation (4.9) we split $E_2(j,k)$ as follows:

$$\begin{aligned}
E_2(j,k) &= \left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \left((\Psi_k A^0)_{2^j}\right)\Lambda^s\left(\left(\psi_j^2\partial_t u\right)_{2^j}\right)\right\rangle_{L^2} \\
&= \left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \left[\left(\Psi_k A^0\right)_{2^j}\Lambda^s\left(\left(\psi_j^2\partial_t u\right)_{2^j}\right) - \Lambda^s\left(\left(\Psi_k A^0\right)_{2^j}\left(\psi_j^2\partial_t u\right)_{2^j}\right)\right]\right\rangle_{L^2} \\
&+ \left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \Lambda^s\left(\left(\Psi_k A^0\right)_{2^j}\left(\psi_j^2\partial_t u\right)_{2^j}\right)\right\rangle_{L^2} \\
&= E_3(j,k) + E_4(j,k).
\end{aligned}$$

In the estimation of the first term $E_3(j,k)$, the Kato-Ponce commutator estimate (4.11) is being used again:

$$\begin{aligned}
&|E_3(j,k)| \\
\le\ & C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}\left\{\left\|\nabla\left(\Psi_k A^0\right)_{2^j}\right\|_{L^\infty}\left\|\left(\psi_j^2\partial_t u\right)_{2^j}\right\|_{H^{s-1}} + \left\|\left(\Psi_k A^0\right)_{2^j}\right\|_{H^s}\left\|\left(\psi_j^2\partial_t u\right)_{2^j}\right\|_{L^\infty}\right\}.
\end{aligned}$$

From (4.16) and the embedding (4.20), we have

$$\begin{aligned}
\left\|\nabla\left(\Psi_k A^0\right)_{2^j}\right\|_{L^\infty} &= 2^j\left\|\left(\nabla\left(\Psi_k A^0 - I\right)\right)_{2^j}\right\|_{L^\infty} + 2^j\left\|\nabla(\Psi_k)_{2^j}\right\|_{L^\infty} \\
&\le C\left\{2^j\left\|\left(\nabla\Psi_k\left(A^0 - I\right)\right)_{2^j}\right\|_{H^{s-1}} + 1\right\}
\end{aligned}$$

and from (4.19)

$$\left\|\left(\Psi_k A^0\right)_{2^j}\right\|_{H^s} \le \left\|\left(\Psi_k\left(A^0 - I\right)\right)_{2^j}\right\|_{H^s} + \left\|\nabla\left(\Psi_k\right)_{2^j}\right\|_{H^s} \le \left\|\left(\Psi_k\left(A^0 - I\right)\right)_{2^j}\right\|_{H^s} + C.$$

Thus

$$\begin{aligned}
&|E_3(j,k)| \\
\le\ & C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}\left\|\left(\psi_j^2\partial_t u\right)_{2^j}\right\|_{H^{s-1}}\left\{2^j\left\|\left(\nabla\Psi_k\left(A^0 - I\right)\right)_{2^j}\right\|_{H^{s-1}} + 1\right\} \\
+\ & C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}\left\|\left(\psi_j^2\partial_t u\right)_{2^j}\right\|_{L^\infty}\left\{\left\|\left(\Psi_k\left(A^0 - I\right)\right)_{2^j}\right\|_{H^s} + 1\right\} \\
\le\ & C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}\left\|\left(\psi_j^2\partial_t u\right)_{2^j}\right\|_{H^{s-1}}\left\{2^j\left\|\left(\nabla\Psi_k\left(A^0 - I\right)\right)_{2^j}\right\|_{H^{s-1}} + 1\right\} \\
+\ & C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}\left\|\left(\Psi_k\left(A^0 - I\right)\right)_{2^j}\right\|_{H^s}\left\|\partial_t u\right\|_{L^\infty} \\
+\ & C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}\left\|\left(\psi_j^2\partial_t u\right)_{2^j}\right\|_{H^{s-1}}.
\end{aligned} \tag{4.21}$$

Now equation (4.9) is being utilized and

$$\begin{aligned}
E_4(j,k) &= \left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \Lambda^s\left(\left(\Psi_k\psi_j^2\right)_{2^j}\left(A^0\partial_t u\right)_{2^j}\right)\right\rangle_{L^2} \\
&= \left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \Lambda^s\left(\left(\Psi_k\psi_j^2\right)_{2^j}\left(\sum_{a=1}^3 A^a\partial_a u\right)_{2^j}\right)\right\rangle_{L^2} \\
&= \sum_{a=1}^3\left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \Lambda^s\left(\left(\Psi_k A^a\right)_{2^j}\left(\psi_j^2\partial_a u\right)_{2^j}\right)\right\rangle_{L^2} \\
&= \sum_{a=1}^3\left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \left[\Lambda^s\left(\left(\Psi_k A^a\right)_{2^j}\left(\psi_j^2\partial_a u\right)_{2^j}\right) - \left(\Psi_k A^a\right)_{2^j}\Lambda^s\left(\left(\psi_j^2\partial_a u\right)_{2^j}\right)\right]\right\rangle_{L^2} \\
&+ \sum_{a=1}^3\left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \left[\left(\Psi_k A^a\right)_{2^j}\Lambda^s\left(\left(\psi_j^2\partial_a u\right)_{2^j}\right)\right]\right\rangle_{L^2} \\
&= E_5(j,k,a) + E_6(j,k,a).
\end{aligned} \tag{4.22}$$

Again, by Kato-Ponce commutator estimate (4.11),

$$
\begin{aligned}
& |E_5(j,k,a)| \\
\leq\ & C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}\left\{\left\|\nabla\left(\Psi_k A^a\right)_{2^j}\right\|_{L^\infty}\left\|\left(\psi_j^2\partial_a u\right)_{2^j}\right\|_{H^{s-1}}+\left\|\left(\Psi_k A^a\right)_{2^j}\right\|_{H^s}\left\|\left(\psi_j^2\partial_a u\right)_{2^j}\right\|_{L^\infty}\right\} \\
\leq\ & C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}\left\{\left\|\nabla A^a\right\|_{L^\infty}+\left\|A^a\right\|_{L^\infty}\right\}2^j\left\|\left(\psi_j^2\partial_a u\right)_{2^j}\right\|_{H^{s-1}} \\
+\ & C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}\left\|\left(\Psi_k A^a\right)_{2^j}\right\|_{H^s}\quad\left\|\partial_a u\right\|_{L^\infty}.
\end{aligned}
\tag{4.23}
$$

Using the commutation $\partial_a\Lambda^s=\Lambda^s\partial_a$ and the fact that $\Lambda^s\left(\psi_j^2 u\right)$ is rapidly decreasing, we see that

$$
\begin{aligned}
0 &= \int\partial_a\left\{\left[\Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right)\right]^T\left(\Psi_k A^a\right)_{2^j}\left[\Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right)\right]\right\}dx \\
&= 2^j\int\left\{\left[\Lambda^s\left(\left(\psi_j^2\partial_a u\right)_{2^j}\right)\right]^T\left(\Psi_k A^a\right)_{2^j}\left[\Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right)\right]\right\}dx \\
&+ 2^j\int\left\{\left[\Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right)\right]^T\left(\Psi_k A^a\right)_{2^j}\left[\Lambda^s\left(\left(\psi_j^2\partial_a u\right)_{2^j}\right)\right]\right\}dx \\
&+ 2^j 2\int\left\{\left[\Lambda^s\left(\left(\left(\partial_a\psi_j\right)\psi_j u\right)_{2^j}\right)\right]^T\left(\Psi_k A^a\right)_{2^j}\left[\Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right)\right]\right\}dx \\
&+ 2^j 2\int\left\{\left[\Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right)\right]^T\left(\Psi_k A^a\right)_{2^j}\left[\Lambda^s\left(\left(\left(\partial_a\psi_j\right)\psi_j u\right)_{2^j}\right)\right]\right\}dx \\
&+ 2^j\int\left\{\left[\Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right)\right]^T\left(\partial_a\left(\Psi_k A^a\right)\right)_{2^j}\left[\Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right)\right]\right\}dx.
\end{aligned}
$$

Now $E_6(j,k,l)$ is equal to the second term, but since $A^a$ is a symmetric matrix, the first and the second terms are equal and also the third and the forth. Hence by (4.17) and Cauchy Schwarz inequality,

$$
\begin{aligned}
|2E_6(j,k,a)| &\leq 2\left\|\left(\Psi_k A^a\right)_{2^j}\right\|_{L^\infty}\left\|\left(\partial_a\psi_j\psi_j u\right)_{2^j}\right\|_{H^s}\quad\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s} \\
&+ \left\|\left(\partial_a\left(\Psi_k A^a\right)\right)_{2^j}\right\|_{L^\infty}\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}^2 \\
&\leq C\left\|A^a\right\|_{L^\infty}\left\|\left(\psi_j u\right)_{2^j}\right\|_{H^s}\quad\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s} \\
&+ \left\{\left\|\partial_a A^a\right\|_{L^\infty}+C\left\|A^a\right\|_{L^\infty}\right\}\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}^2.
\end{aligned}
$$

Taking the sum $\sum 2^{(\frac{3}{2}+\delta)2j}E(j)$ we are coming across three types of summations:

1. Given $v\in H_{s_1,\delta}$, $w\in H_{s_2,\delta}$ and $\gamma_i$ equals 1 or 2, then

$$
\begin{aligned}
& \sum_{j=0}^{\infty}2^{(\frac{3}{2}+\delta)2j}\left\|\left(\psi_j^{\gamma_1}v\right)_{2^j}\right\|_{H^{s_1}}\left\|\left(\psi_j^{\gamma_2}w\right)_{2^j}\right\|_{H^{s_2}} \\
\leq\ & \frac{1}{2}\left(\sum_{j=0}^{\infty}2^{(\frac{3}{2}+\delta)2j}\left\|\left(\psi_j^{\gamma_1}v\right)_{2^j}\right\|_{H^{s_1}}^2+2^{(\frac{3}{2}+\delta)2j}\left\|\left(\psi_j^{\gamma_2}w\right)_{2^j}\right\|_{H^{s_2}}^2\right) \\
\leq\ & C\left(\|v\|_{H_{s_1,\delta}}^2+\|w\|_{H_{s_2,\delta}}^2\right),
\end{aligned}
$$

where in the last inequality the equivalence of the norms, (see Corollary 5.6), was involved.

2. Given $v \in H_{s,\delta}$ and $w \in H_{s,\delta}$, then from the scaling property (4.18) and (4.17) we have

$$
\begin{aligned}
& \sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j} \left\|(\psi_j^2 v)_{2^j}\right\|_{H^s} \|(\Psi_k w)_{2^j}\|_{H^s} \\
\leq \;& \frac{1}{2}\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j} \left\|(\psi_j^2 v)_{2^j}\right\|_{H^s}^2 + \frac{1}{2}\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j} \|(\Psi_k w)_{2^j}\|_{H^s}^2 \\
\leq \;& \frac{7}{2}\|v\|_{H_{s,\delta}}^2 + C\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j} \|(\Psi_k w)_{2^k}\|_{H^s}^2 \\
\leq \;& \frac{7}{2}\|v\|_{H_{s,\delta}}^2 + C\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2k} \|(\psi_k w)_{2^k}\|_{H^s}^2 \\
\leq \;& C\left(7\|v\|_{H_{s,\delta}}^2 + 7\|w\|_{H_{s,\delta}}^2\right).
\end{aligned}
$$

3. Given $v \in H_{s_1,\delta}$, $w \in H_{s_2,\delta}$, $z \in H_{s_3,\delta}$ and $\gamma_i$ equals 1 or 2, then by Hölder inequality, Corollary 6.2 and the same arguments as in type 2, we get

$$
\begin{aligned}
& \sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j} \left\|(\psi_j^{\gamma_1} v)_{2^j}\right\|_{H^{s_1}} \left\|(\psi_j^{\gamma_2} w)_{2^j}\right\|_{H^{s_2}} 2^j \|(\nabla(\Psi_k z))_{2^j}\|_{H^{s_3-1}} \\
\leq \;& \sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)j} \left\|(\psi_j^{\gamma_1} v)_{2^j}\right\|_{H^{s_1}} 2^{(\frac{3}{2}+\delta)j} \left\|(\psi_j^{\gamma_2} w)_{2^j}\right\|_{H^{s_2}} 2^{(\frac{3}{2}+\delta+1)j} \|(\nabla(\Psi_k z))_{2^j}\|_{H^{s_3-1}} \\
\leq \;& \left(\left(\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} \left(2^{\left(\frac{3}{2}+\delta\right)2j} \left\|\left(\psi_j^{\gamma_1} v\right)_{2^j}\right\|_{H^{s_1}}^2\right)^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
\times \;& \left(\left(\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} \left(2^{\left(\frac{3}{2}+\delta\right)2j} \left\|\left(\psi_j^{\gamma_2} w\right)_{2^j}\right\|_{H^{s_2}}^2\right)^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
\times \;& C\left(\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{\left(\frac{3}{2}+\delta+1\right)2j} \|(\nabla(\psi_k z))_{2^k}\|_{H^{s_3-1}}^2\right)^{\frac{1}{2}} \\
\leq \;& \left(\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|\left(\psi_j^{\gamma_1} v\right)_{2^j}\right\|_{H^{s_1}}^2\right)^{\frac{1}{2}} \\
\times \;& \left(\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|\left(\psi_j^{\gamma_2} w\right)_{2^j}\right\|_{H^{s_2}}^2\right)^{\frac{1}{2}} \\
\times \;& C\left(\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{\left(\frac{3}{2}+\delta+1\right)2k} \|(\nabla(\psi_k z))_{2^k}\|_{H^{s_3-1}}^2\right)^{\frac{1}{2}} \\
\leq \;& C\|v\|_{H_{s_1,\delta}} \quad \|w\|_{H_{s_2,\delta}} \quad \|\nabla z\|_{H_{s_3-1,\delta+1}} \\
\leq \;& C\|v\|_{H_{s_1,\delta}} \quad \|w\|_{H_{s_2,\delta}} \quad \|z\|_{H_{s_3,\delta}} \\
\leq \;& C\left(\|v\|_{H_{s_1,\delta}}^2 + \left(\|w\|_{H_{s_2,\delta}}\|z\|_{H_{s_3,\delta}}\right)^2\right).
\end{aligned}
$$

Applying these three types of inequalities we have,

$$\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j}|2E_6(j,k,a)| \leq C\left(\|A^a\|_{L^\infty} + \|\partial_a A^a\|_{L^\infty}\right)\|u\|^2_{H_{s,\delta}}, \tag{4.24}$$

$$\begin{aligned}
&\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j}|E_5(j,k,a)| \\
\leq\ & C\left(\|\nabla A^a\|_{L^\infty} + \|A^a\|_{L^\infty}\right)\left\{\|u\|^2_{H_{s,\delta}} + \|\partial_a u\|^2_{H_{s-1,\delta+1}}\right\} + C\left\{\|u\|^2_{H_{s,\delta}} + \|A^a\|^2_{H_{s,\delta}}\|\partial_a u\|^2_{L^\infty}\right\} \\
\leq\ & C\left(\|\nabla A^a\|_{L^\infty} + \|A^a\|_{L^\infty}\right)\left\{\|u\|^2_{H_{s,\delta}} + \|u\|^2_{H_{s,\delta}}\right\} + C\left\{\|u\|^2_{H_{s,\delta}} + \|A^a\|^2_{H_{s,\delta}}\|\partial_a u\|^2_{H_{s-1,\delta+1}}\right\} \\
\leq\ & C\left\{2\|\nabla A^a\|_{L^\infty} + 2\|A^a\|_{L^\infty} + \|A^a\|^2_{H_{s,\delta}} + 1\right\}\|u\|^2_{H_{s,\delta}},
\end{aligned} \tag{4.25}$$

here we have applied Corollary 6.2 and Theorem 6.9 to $\partial_a u$. Applying again Theorem 6.9 to $\|\partial_t u\|_{L^\infty}$ we have

$$\begin{aligned}
&\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j}|E_3(j,k)| \\
\leq\ & C\left\{\|u\|^2_{H_{s,\delta}} + \|\partial_t u\|^2_{H_{s-1,\delta}}\left\|\nabla\left(A^0 - I\right)\right\|^2_{H_{s-1,\delta+1}}\right\} + 2C\left\{\|u\|^2_{H_{s,\delta}} + \|\partial_t u\|^2_{H_{s-1,\delta}}\right\} \\
+\ & C\left\{\|u\|^2_{H_{s,\delta}} + \left\|\left(A^0 - I\right)\right\|^2_{H_{s,\delta}}\|\partial_t u\|^2_{L^\infty}\right\} \\
\leq\ & 2C\left\{\|u\|^2_{H_{s,\delta}} + \|\partial_t u\|^2_{H_{s-1,\delta}}\left(1 + \left\|A^0 - I\right\|^2_{H_{s,\delta}}\right)\right\}
\end{aligned} \tag{4.26}$$

and finally

$$\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}|E_1(j)| \leq C\left\|A^0\right\|_{L^\infty}\left\{\|u\|^2_{H_{s,\delta}} + \|\partial_t u\|^2_{H_{s-1,\delta}}\right\}. \tag{4.27}$$

Recalling that

$$\langle u, \partial_t u\rangle_{s,\delta,A^0} = \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}E(j) = \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\left\langle\Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \left((A^0)_{2^j}\right)\Lambda^s\left(\left(\psi_j^2\partial_t u\right)_{2^j}\right)\right\rangle_{L^2},$$

then inequalities (4.24), (4.25), (4.27) and (4.27) imply that

$$\langle u, \partial_t u\rangle_{s,\delta,A^0} \leq C\left(\|A^\alpha\|_{L^\infty}, \|\nabla A^a\|_{L^\infty}, \|A^a\|_{H_{s,\delta}}, \|A^0 - I\|_{H_{s,\delta}}, \|\partial_t u\|_{H_{s-1,\delta}}\right)\left\{\|u\|^2_{H_{s,\delta}} + 1\right\}.$$

Since $s > \frac{5}{2}$ and $\delta \geq -\frac{3}{2}$ we can use Theorem 6.9 (of the Appendix 6) and bound the norms $\|A^\alpha\|_{L^\infty}$ and $\|\nabla A^\alpha\|_{L^\infty}$ by the norms $\|A^0 - I\|_{H_{s,\delta}}$ and $\|A^a\|_{H_{s,\delta}}$. Thus, combining these bounds with above inequality and inequality (4.12), we have obtained

$$\frac{d}{dt}\langle u(t), u(t)\rangle_{s,\delta,A^0} \leq C\left(\|u(t)\|^2_{H_{s,\delta}} + 1\right), \tag{4.28}$$

where $C = C(\|A^a\|_{H_{s,\delta}}, \|A^0 - I\|_{H_{s,\delta}}, \|\partial_t u\|_{H_{s-1,\delta}}, \|\partial_t A^0\|_{L^\infty})$. Inserting the equivalence of norms $\|u\|^2_{H_{s,\delta}} \leq \mu\|u\|^2_{H_{s,\delta,A^0}}$ in (4.28), we obtain (4.10) which completes the proof of Lemma 4.4. ∎

We may extend the energy estimate (4.10) to a non-homogeneous symmetric hyperbolic systems.

**Lemma 4.6** ***(An energy estimate)*** *Let $s > \frac{5}{2}$, $\delta \geq -\frac{3}{2}$, $A^\alpha = A^\alpha(t,x)$ be $m \times m$ symmetric matrices such that $(A^0(t,\cdot) - I), A^a(t,\cdot) \in H_{s,\delta}$ and $A^0$ satisfies (4.5). Let $B(t,\cdot)$, $F(t,\cdot) \in H_{s,\delta}$. If $u(t,\cdot)$ is a $C_0^\infty$ solution of the linear hyperbolic system*

$$A^0(t,x)\partial_t u = \sum_{a=1}^{3} A^a(t,x)\partial_a u + B(t,x)u + F(t,x), \tag{4.29}$$

*then*

$$\frac{d}{dt}\|u(t)\|^2_{H_{s,\delta,A^0}} \leq C\left(\mu\|u(t)\|^2_{H_{s,\delta,A^0}} + 1\right), \tag{4.30}$$

*where the constant $C$ depends on $\|A^a\|_{H_{s,\delta}}$, $\|A^0 - I\|_{H_{s,\delta}}$, $\|\partial_t u\|_{H_{s-1,\delta}}$, $\|\partial_t A^0\|_{L^\infty}$, $\|B\|_{H_{s,\delta}}$ and $\|F\|_{H_{s,\delta}}$.*

**Proof (of Lemma 4.6)** This proof is precisely as the previous one expect the two terms

$$\left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \Lambda^s\left(\left(\Psi_k B\right)_{2^j}\left(\psi_j^2 u\right)_{2^j}\right)\right\rangle_{L^2} \tag{4.31}$$

and

$$\left\langle \Lambda^s\left(\left(\psi_j^2 u\right)_{2^j}\right), \Lambda^s\left(\left(\Psi_k \psi_j^2 F\right)_{2^j}\right)\right\rangle_{L^2} \tag{4.32}$$

which are added to (4.22). Using the algebra properties of $H^s$ spaces, we see that (4.31) is less than

$$C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|^2_{H^s}\|(\Psi_k B)_{2^j}\|_{H^s} \leq C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|^2_{H^s}\|B\|_{H_{s,\delta}};$$

and by Cauchy Schwarz inequality (4.32) is less than

$$C\left\|\left(\psi_j^2 u\right)_{2^j}\right\|_{H^s}\left\|\left(\Psi_k\psi_j^2 F\right)_{2^j}\right\|_{H^s} \leq C\frac{1}{2}\left\{\left\|\left(\psi_j^2 u\right)_{2^j}\right\|^2_{H^s} + \left\|\left(\psi_j^2 F\right)_{2^j}\right\|^2_{H^s}\right\}.$$

Multiplying (4.31) and (4.32) by $2^{(\frac{3}{2}+\delta)2j}$ and taking the sum, it results with two quantities less than $\|u\|^2_{H_{s,\delta}}\|B\|_{H_{s,\delta}}$ and $\left(\|u\|^2_{H_{s,\delta}} + \|F\|^2_{H_{s,\delta}}\right)$ respectively. ■

## 4.3 Construction of the iteration

We assume $u_0(x)$, the initial value of (4.1), is contained in $G_1$, where the origin belongs to $G_1$ and $G_1$ is a compact subset of an open set $G$ of $\mathbb{R}^m$. In addition we assume,

$$\frac{1}{\mu}U^T U \leq U^T A^0(U;t,x)U \leq \mu U^T U \qquad \text{for all} \quad U \in G_2, \tag{4.33}$$

where $G_2$ is a compact set of $G$ such that $G_1 \Subset G_2$ and $\mu > 0$.

**Remark 4.7** *Since the matrix $A^0$ is continuous, the initial condition (4.2) guarantees the existence of a domain $G_2$.*

The initial data $u_0$ will be approximated by a sequence $\{u_0^k\}$ of smooth functions with compact support, which converges to $u_0$ in $H_{s,\delta}(\mathbb{R}^3)$. It follows from the embedding $\|v\|_{L^\infty} \leq C\|v\|_{H_{s,\delta}}$ (see Theorem 6.9, and the density Theorem 6.10, that there is a positive $R$, $u_0^0 \in C_0^\infty(\mathbb{R}^3)$ and $\{u_0^k\}_{k=1}^\infty \subset C_0^\infty(\mathbb{R}^3)$ such that

$$\|u_0^0\|_{H_{s+1,\delta}} \leq C\|u_0\|_{H_{s,\delta}}, \tag{4.34}$$

$$\|u_0^0 - u_0\|_{H_{s,\delta}} \leq \frac{R}{\mu 8}, \tag{4.35}$$

$$\|u - u_0^0\|_{H_{s,\delta}} \leq R \Rightarrow u \in G_2 \tag{4.36}$$

and

$$\|u_0^k - u_0\|_{H_{s,\delta}} \leq 2^{-k}\frac{R}{\mu 8}. \tag{4.37}$$

The iteration procedure is defined as follows: $u^0(t,x) = u_0^0(x)$ and $u^{k+1}(t,x)$ is a solution to the linear initial value problem

$$\begin{cases} A^0(u^k;t,x)\partial_t u^{k+1} = \sum_{a=1}^3 A^a(u^k;t,x)\partial_a u^{k+1} + B(u^k;t,x)u^{k+1} + F(u^k;t,x), \\ u^{k+1}(0,x) = u_0^{k+1}(x). \end{cases} \tag{4.38}$$

The existence of $\{u^k(t,x)\} \subset C_0^\infty(\mathbb{R}^3)$ follows from:

**Theorem 4.8** ***(Existence of classical solutions of a linear symmetric hyperbolic system)*** *Let $A^\alpha$, $B$ and $F$ be $C^\infty$ functions and $v_0 \in C_0^\infty(\mathbb{R}^3)$ be an initial datum. Then the linear system*

$$\begin{cases} A^0(t,x)\partial_t v = \sum_{a=1}^3 A^a(t,x)\partial_a v + B(t,x)v + F(t,x) \\ v(0,x) = v_0(x) \end{cases} \tag{4.39}$$

*has a unique solution $v(t,x)$ such that $v(t,x) \in C^\infty$ and it has compact support in $\mathbb{R}^3$ for each fixed $t$.*

For the proof we refer to John [14]. It is evident from Theorem 4.8 and inequalities (4.33) and (4.36) that for each $k$: $u^k(t,x)$ is well defined, $u^k(t,x) \in C^\infty$, $u^k(t,x)$ has compact support in $\mathbb{R}^3$ and $u^k(t,x) \in G_2$ for some positive $T$. We put

$$T_k = \sup\{T : \sup_{0<t<T} \|u^k(t) - u_0^0\|_{H_{s,\delta}} \leq R\}. \tag{4.40}$$

Our next issue is to show the existence of $T^* > 0$ such that $T_k \geq T^*$ for $k = 1,2,3,\ldots$

## 4.4 Boundedness in the $H_{s,\delta}$ norm

We introduce the following notations: $u(t) := u(t,x)$ and

$$|||u|||_{s,\delta,T} := \sup\{\|u(t)\|_{H_{s,\delta}} : 0 \leq t \leq T\}. \tag{4.41}$$

The main result of this subsection is:

**Lemma 4.9** ***(Boundedness in the $H_{s,\delta}$ norm)*** *There are positive constants $T^*$ and $L$ such that*

(A) $|||u^k - u_0^0|||_{s,\delta,T^*} \leq R$

(B) $|||\partial_t u^k|||_{s-1,\delta+1,T^*} \leq L$.

**Proof (of Lemma 4.9)** We first prove $(B)$. Let

$$G^{k+1} = \sum_{a=1}^3 A^a(u^k;t,x)\partial_a u^{k+1} + B(u^k;t,x)u^{k+1} + F(u^k;t,x),$$

then by Proposition 6.4 and Moser type estimate, Theorem 6.6 with Remark 6.7,

$$\begin{aligned}
& \|G^{k+1}\|_{H_{s-1,\delta+1}} \\
\leq & \sum_{a=1}^{3}\|A^a(u^k)\|_{H_{s,\delta}}\|\partial_a u^k\|_{H_{s-1,\delta+1}}+\|B(u^k)\|_{H_{s,\delta}}\|u^k\|_{H_{s,\delta}}+\|F(u^k)\|_{H_{s,\delta}} \\
\leq & \sum_{a=1}^{3}\left(C\|u^k\|_{H_{s,\delta}}+\|A^a(0)\|_{H_{s,\delta}}\right)\|u^k\|_{H_{s,\delta}}+\left(C\|u^k\|_{H_{s,\delta}}+\|B(0)\|_{H_{s,\delta}}\right)\|u^k\|_{H_{s,\delta}} \\
+ & C\|u^k\|_{H_{s,\delta}}+\|F(0)\|_{H_{s,\delta}}.
\end{aligned} \tag{4.42}$$

The constant $C$ here depends on $\|A^a\|_{C^{N+1}(G_2)}$, $\|B\|_{C^{N+1}(G_2)}$, $\|F\|_{C^{N+1}(G_2)}$ and $\|u^k\|_{L^\infty}$ (see (6.7)). Since

$$\|u^k(t)\|_{H_{s,\delta}}\leq\|u^k(t)-u_0^0\|_{H_{s,\delta}}+\|u_0^0\|_{H_{s,\delta}}, \tag{4.43}$$

the induction assumption (A) and inequality (4.34) imply that $\|u^k\|_{H_{s,\delta}}\leq R+C\|u_0\|_{H_{s,\delta}}$. Using the embedding $\|u^k\|_{L^\infty}\leq C\|u^k\|_{H_{s,\delta}}$, we see that $\|G^{k+1}\|_{H_{s-1,\delta+1}}\leq C_1(R)$, where the constant $C_1(R)$ depends upon $R$, condition (H3) and the initial data, but it is independent of $k$. From (4.38) we have

$$\partial_t u^{k+1}=\left(A^0(u^k;t,x)\right)^{-1}G^{k+1}=\left(\left(A^0(u^k;t,x)\right)^{-1}-I\right)G^{k+1}+G^{k+1}.$$

Repeating same arguments as above, we conclude that

$$\|\left(\left(A^0(u^k;t,x)\right)^{-1}-I\right)G^{k+1}\|_{H_{s-1,\delta+1}}\leq C_2(R)$$

and the constant $C_2(R)$ does not depend on $k$. We take $L=C_1(R)+C_2(R)$. Here we have used Moser estimate with $F(u)=A^{-1}(u)-I$, and the formula $\frac{\partial A^{-1}(u)}{\partial u}=A^{-1}(u)\frac{\partial A(u)}{\partial u}A^{-1}(u)$. Thus the constant $C(R)$ depends on $\|A^0\|_{C^{N+2}(G_2)}$ and $\mu$.

We turn now to show (A). Let $V^{k+1}=u^{k+1}-u_0^0$, then inserting it in the equation (4.38) we have obtained

$$\begin{aligned}
A^0(u^k;t,x)\partial_t V^{k+1} &= A^0(u^k;t,x)u_t^{k+1}=\sum_{a=1}^{3}A^a(u^k;t,x)\partial_a u^{k+1}+B(u^k;t,x)u^k+F(u^k;t,x) \\
&= \sum_{a=1}^{3}A^a(u^k;t,x)\partial_a V^{k+1}+B(u^k;t,x)V^{k+1}+F(u^k;t,x) \\
&+ \sum_{a=1}^{3}A^a(u^k;t,x)\partial_a u_0^0+B(u^k;t,x)u_0^0
\end{aligned} \tag{4.44}$$

and $V^{k+1}(0,x)=u_0^{k+1}(0,x)-u_0^0(x)$. At this stage we would like employ the energy estimate Lemma 4.6. Due the fact that the coefficients of (4.44) depend on $u^k$, it is obligatory to control the constant of (4.30) in terms of $\|u^k\|_{H_{s,\delta}}$. Therefore we need to bound $\|\left(A^0(u^k;t,x)-I\right)\|_{H_{s,\delta}}$, $\|A^a(u^k;t,x)\|_{H_{s,\delta}}$, $\|B(u^k;t,x)\|_{H_{s,\delta}}$, $\|F(u^k;t,x)\|_{H_{s,\delta}}$ and $\|\frac{\partial}{\partial t}A^0(u^k;t,x)\|_{L^\infty}$ by $\|u^k\|_{H_{s,\delta}}$. The first four are similar, so take for example $A^a(u^k;t,x)$: We use assumption (H2), Moser type estimate, Theorem 6.6 with Remark 6.7 and get that

$$\|A^a(u^k;t,x)\|_{H_{s,\delta}}\leq C\left\{\|A^a\|_{C^{N+1}(G_2)}\left(1+\|u^k\|_{L^\infty}^N\right)\right\}\|u^k\|_{H_{s,\delta}}+\|A^a(0;t,\cdot)\|_{H_{s,\delta}}. \tag{4.45}$$

For the last one we have

$$\begin{aligned}
\|\frac{\partial}{\partial t}A^0(u^k;t,x)\|_{L^\infty} &= \|\frac{\partial}{\partial u}A^0(u^k;t,x)\partial_t u^k(t,x)+\partial_t A^0(u^k;t,x)\|_{L^\infty} \\
&\leq \|\frac{\partial}{\partial u}A^0(u^k;t,x)\|_{L^\infty}\|\partial_t u^k(t,x)\|_{L^\infty}+\|\partial_t A^0(u^k;t,x)\|_{L^\infty} \\
&\leq C\|\frac{\partial}{\partial u}A^0(u^k;t,x)\|_{L^\infty}\|\partial_t u^k(t,x)\|_{H_{s-1,\delta+1}}+\|\partial_t A^0(u^k;t,x)\|_{L^\infty}.
\end{aligned} \tag{4.46}$$

We conclude from inequalities (4.45) and (4.46), Theorem 6.9, the inductions hypothesis (A) and (B), (4.36) and (H4) that the constant of (4.30) depends on $R$, $L$, $\|u_0\|_{H_{s,\delta}}$ and the $H_{s,\delta}$-norm of the coefficients, but it is independent of $k$. Hence, the energy estimate Lemma 4.6 implies that

$$\frac{d}{dt}\|V^{k+1}(t)\|^2_{H_{s,\delta,A^0}} \leq C(R,L)\left(\mu\|V^{k+1}(t)\|^2_{H_{s,\delta,A^0}}+1\right), \tag{4.47}$$

Applying Gronwall's inequality, (4.35), (4.37) and the equivalence (4.8) results in

$$\begin{aligned}
|||V^{k+1}|||^2_{s,\delta,T} &\leq \mu e^{C(R,L)\mu T}\left(\mu\|V^{k+1}(0)\|^2_{H_{s,\delta}}+C(R,L)T\right)\\
&= \mu e^{C(R,L)\mu T}\left(\mu\|u_0^{k+1}-u_0^0\|^2_{H_{s,\delta}}+C(R,L)T\right)\\
&\leq \mu e^{C(R,L)\mu T}\left(\mu\left(\|u_0^{k+1}-u_0\|^2_{H_{s,\delta}}+\|u_0^0-u_0\|^2_{H_{s,\delta}}\right)+C(R,L)T\right)\\
&\leq e^{C(R,L)\mu T}\left(2\mu^2\left(\frac{R}{\mu 8}\right)^2+\mu C(R,L)T\right).
\end{aligned} \tag{4.48}$$

Therefore $|||V^{k+1}|||^2_{s,\delta,T} \leq R^2$, if $\mu C(R,L)T \leq \min\{\log 2, \frac{15}{32}R^2\}$. Thus taking $T^* = (\mu C(R,L))^{-1}\min\{\log 2, \frac{15}{32}R^2\}$ proves $(A)$ and completes the proof of Lemma 4.9. ∎

## 4.5 Contraction in the lower norm

We show here that $\{u^k\}$ has a contraction property in $\|\cdot\|_{0,\delta,T^{**}}$ for a positive $T^{**}$ (see (4.41)). In order to achieved it we need an energy estimate in $H_{0,\delta} \backsim L^2_\delta$. For that purpose we introduce the below inner-product in $L^2_\delta$: for two vectors $u$ and $v$ in $L^2_\delta$, we set

$$\langle u,v\rangle_{L^2_\delta,A^0} = \int (1+|x|)^{2\delta}\left(u^TA^0v\right)dx, \tag{4.49}$$

and its associated norm $\|u\|^2_{L^2_\delta,A^0} = \langle u,u\rangle_{L^2_\delta,A^0}$. The ordinary norm is denoted by $\|u\|^2_{L^2_\delta} = \langle u,u\rangle_{L^2_\delta,I}$. Since $A^0$ satisfies (4.33),

$$\frac{1}{\mu}\|u\|^2_{L_\delta} \leq \langle u,u\rangle_{L^2_\delta,A^0} \leq \mu\|u\|^2_{L_\delta}, \tag{4.50}$$

and hence by Theorem we obtain 5.2, $\|u\|^2_{L^2_\delta,A^0} \simeq \|u\|_{H_{0,\delta}}$.

**Proposition 4.10** ***(Energy estimate in $L^2_\delta$)*** *Suppose $u$ satisfies the linear hyperbolic system (4.39), then*

$$\frac{d}{dt}\langle u(t),u(t)\rangle_{L^2_\delta,A^0} \leq \mu C\langle u(t),u(t)\rangle_{L^2_\delta,A^0}+\|F\|^2_{L^2_\delta}, \tag{4.51}$$

*where $C = C(\|\partial_tA^0\|_{L^\infty},\|A^a\|_{L^\infty},\|B\|_{L^\infty},\|\partial_aA^a\|_{L^\infty})$.*

**Proof (of Proposition 4.10)** Taking the derivative of (4.49) with respect to $t$, we get

$$\begin{aligned}
\frac{d}{dt}\langle u,u\rangle_{L^2_\delta,A^0} &= 2\langle u,\partial_tu\rangle_{L^2_\delta,A^0}+\int(1+|x|)^{2\delta}\left(u^T\partial_tA^0u\right)dx\\
&= 2\sum_{a=1}^3\int(1+|x|)^{2\delta}\left(u^TA^a\partial_au\right)dx+2\int(1+|x|)^{2\delta}\left(u^TBu\right)dx\\
&+ 2\int(1+|x|)^{2\delta}\left(u^TF\right)dx+\int(1+|x|)^{2\delta}\left(u^T\partial_tA^0u\right)dx\\
&= 2\sum_{a=1}^3L_{1,a}+2L_2+2L_3+L_4.
\end{aligned}$$

Clearly,

$$|L_2| \leq \|B\|_{L^\infty} \int (1+|x|)^{2\delta} |u|^2 dx \leq \|B\|_{L^\infty} \|u\|^2_{L^2_\delta}$$

and in a similar way we obtain the estimates of $L_4$ while by Cauchy-Schwarz inequality,

$$|L_3| \leq \|u\|_{L^2_\delta} \|F\|_{L^2_\delta} \leq \frac{1}{2}\left(\|u\|^2_{L^2_\delta} + \|F\|^2_{L^2_\delta}\right).$$

Now,

$$\begin{aligned}
0 &= \int \partial_a \left((1+|x|)^{2\delta} \left(u^T A^a u\right)\right) dx \\
&= 2\delta \int (1+|x|)^{2\delta-1} \frac{x_a}{|x|} \left(u^T A^a u\right) dx + \int (1+|x|)^{2\delta} \left((\partial_a u)^T A^a u\right) dx \\
&+ \int (1+|x|)^{2\delta} \left(u^T \partial_a A^a u\right) dx + \int (1+|x|)^{2\delta} \left(u^T A^a \partial_a u\right) dx,
\end{aligned}$$

and since $A^0$ is symmetric, the second and the fourth terms are equal to $L_{1,a}$. Hence,

$$\begin{aligned}
2|L_{1,a}| &\leq 2\delta \int (1+|x|)^{2\delta} \frac{|A^0|}{1+|x|} \left(|u|^2\right) dx + \int (1+|x|)^{2\delta} |\partial_a A^a| |u|^2 dx \\
&\leq \left(\|A^a\|_{L^\infty} + \|\partial_a A\|_{L^\infty}\right) \|u\|^2_{L^2_\delta}.
\end{aligned}$$

■

In order to proof the contraction we shall also need the following proposition.

**Proposition 4.11** ***(Difference estimate in $L^2_\delta$)*** *Let $G : \mathbb{R}^m \to \mathbb{R}^m$ be a $C^1$ mapping. Then*

$$\|G(u) - G(v)\|^2_{L^2_\delta} \leq \|\nabla G\|^2_{L^\infty} \|u - v\|^2_{L^2_\delta}. \tag{4.52}$$

**Proof (of Proposition 4.11)**

$$\|G(u) - G(v)\|^2_{L^2_\delta} = \int (1+|x|)^{2\delta} (G(u) - G(v))^2 dx$$

$$= \int (1+|x|)^{2\delta} \left(\int_0^1 \nabla G(su + (1-s)v)(u-v) ds\right)^2 dx \leq \|\nabla G\|^2_{L^\infty} \|u - v\|^2_{L^2_\delta}.$$

■

**Lemma 4.12** ***(Contraction in a lower norm)*** *There is a positive $T^{**}$, $0 < \Lambda < 1$ and a positive sequence $\{\beta_k\}$ with $\sum \beta_k < \infty$ such that*

$$|||u^{k+1} - u^k|||_{0,\delta,T^{**}} \leq \Lambda |||u^k - u^{k-1}|||_{0,\delta,T^{**}} + \beta_k. \tag{4.53}$$

*Here* $|||u|||_{0,\delta,T^{**}} = \sup\{\|u(t)\|_{H_{0,\delta}} : 0 \leq t \leq T^{**}\}$.

**Proof (of Lemma 4.12)** Since $u^k$ satisfies equation (4.38), the difference $\left[u^{k+1} - u^k\right]$ will satisfy

$$\begin{aligned}
&A^0(u^k; t, x)\partial_t \left[u^{k+1} - u^k\right] = \sum_{a=1}^{3} A^a(u^k; t, x)\partial_a \left[u^{k+1} - u^k\right] \\
&+ B(u^k; t, x)\left[u^{k+1} - u^k\right] + F^k,
\end{aligned} \tag{4.54}$$

where

$$\begin{aligned}
F^k &= -\left[A^0(u^k; t, x) - A^0(u^{k-1}; t, x)\right]\partial_t u^k + \sum_{a=1}^{3}\left[A^a(u^k; t, x) - A^a(u^{k-1}; t, x)\right]\partial_a u^k \\
&+ \left[B(u^k; t, x) - B(u^{k-1}; t, x)\right] u^k + \left[F(u^k; t, x) - F(u^{k-1}; t, x)\right].
\end{aligned}$$

Applying Proposition 4.10 to equation (4.54) above we have

$$\frac{d}{dt}\langle\left[u^{k+1}-u^{k}\right],\left[u^{k+1}-u^{k}\right]\rangle_{L^2_\delta,A^0}\leq \mu C\langle\left[u^{k+1}-u^{k}\right],\left[u^{k+1}-u^{k}\right]\rangle_{L^2_\delta,A^0}+\|F^k\|^2_{L^2_\delta}. \tag{4.55}$$

Thus Gronwall's inequality yields,

$$\|\left[u^{k+1}(t)-u^{k}(t)\right]\|^2_{L^2_\delta,A^0}\leq e^{\mu Ct}\left[\|\left[u^{k+1}(0)-u^{k}(0)\right]\|^2_{L^2_\delta,A^0}+\int_0^t\|F^k(s)\|_{L^2_\delta}ds\right]. \tag{4.56}$$

The constant $C$ in inequalities (4.55) and (4.56) depends on $\|A^a(u^k;t,x)\|_{L^\infty}$, $\|B(u^k;t,x)\|_{L^\infty}$, $\|\partial_t(A^0(u^k;t,x))\|_{L^\infty}$ and $\|\partial_a\left(A^a(u^k;t,x)\right)\|_{L^\infty}$. The first two of them are bounded by a constant independent of $k$, since it follows from (A) of Lemma 4.9 that $u^k\in G_2$. The estimation of $\|\partial_t(A^0(u^k;t,x))\|_{L^\infty}$ is done in (4.46) and for the last one, since $s-1>\frac{3}{2}$, we can use Theorem 6.9 and Corollary 6.2 and get

$$\begin{aligned}
&\|\partial_a\left(A^a(u^k;t,x)\right)\|_{L^\infty}\leq\|\frac{\partial}{\partial u}A^a(u^k;t,x)\partial_a u^k\|_{L^\infty}+\|\partial_a A^a(u^k;t,x))\|_{L^\infty}\\
&\leq C\|\frac{\partial}{\partial u}A^a(u^k;t,x)\|_{L^\infty}\|\partial_a u^k\|_{H_{s-1,\delta+1}}+\|\partial_a A^a(u^k;t,x))\|_{L^\infty}\\
&\leq C\|\frac{\partial}{\partial u}A^a(u^k;t,x)\|_{L^\infty}\|u^k\|_{H_{s,\delta}}+\|\partial_a A^a(u^k;t,x))\|_{L^\infty}
\end{aligned}$$

Lemma 4.9 (A) implies that $\|u^k\|_{H_{s,\delta}}$ is bounded and $u^k\in G_2$, therefore the above inequality shows that $\|\partial_a\left(A^a(u^k;t,x)\right)\|_{L^\infty}$ is bounded by a constant independent of $k$. From Proposition 4.11 we obtain

$$\begin{aligned}
\left\|F^k\right\|^2_{L^2_\delta}\leq 2\Bigg\{&\|\nabla A^0\|^2_{L^\infty(G_2)}\left\|\partial_t u^k\right\|^2_{L^\infty}+\sum_{a=1}^{3}\|\nabla A^a\|^2_{L^\infty(G_2)}\left\|\partial_a u^k\right\|^2_{L^\infty}\\
&+\|\nabla B\|^2_{L^\infty(G_2)}\left\|u^k\right\|^2_{L^\infty}+\|\nabla F\|^2_{L^\infty(G_2)}\Bigg\}\left\|[u^k-u^{k-1}]\right\|^2_{L^2_\delta},
\end{aligned} \tag{4.57}$$

here $\nabla$ is the gradient with respect to $u$. Since by Theorem 6.9 and Corollary 6.2, $\left\|\partial_t u^k\right\|_{L^\infty}\leq C\left\|\partial_t u^k\right\|_{H_{s-1,\delta+1}}$, $\left\|\partial_a u^k\right\|_{L^\infty}\leq C\left\|u^k\right\|_{H_{s-1,\delta+1}}\leq C\left\|u^k\right\|_{H_{s,\delta}}$ and $\left\|u^k\right\|_{L^\infty}\leq C\left\|u^k\right\|_{H_{s,\delta}}$, it follows from (4.57) and Lemma 4.9 that

$$\left\|F^k(s)\right\|^2_{L^2_\delta}\leq C_1\left\|[u^k(s)-u^{k-1}(s)]\right\|^2_{L^2_\delta}, \tag{4.58}$$

where the constant $C_1$ depends upon $R$ and $L$ of Lemma 4.9, but it is independent of $k$. By the equivalence $\|u\|^2_{L^2_\delta,A^0}\simeq\|u\|_{H_{0,\delta}}$, (4.58) and (4.56) above, we conclude that

$$\begin{aligned}
&\left\|[u^{k+1}(t)-u^k(t)\right\|^2_{H_{0,\delta}}\\
\leq\ &C_2e^{\mu Ct}\left[\left\|[u^{k+1}(0)-u^k(0)]\right\|^2_{H_{0,\delta}}+C_1\int_0^t\left\|[u^k(s)-u^{k-1}(s)\right\|^2_{H_{0,\delta}}\right]\\
\leq\ &C_2e^{\mu Ct}\left[\left\|[u^{k+1}(0)-u^k(0)]\right\|^2_{H_{0,\delta}}+C_1t\sup_{0\leq s\leq t}\left\|[u^k(s)-u^{k-1}(s)\right\|^2_{H_{0,\delta}}\right],
\end{aligned}$$

where $C_1$, $C_2$ and $C$ do not depend on $k$. Hence

$$\begin{aligned}
&|||[u^{k+1}-u^k|||_{0,\delta,T^{**}}\\
\leq\ &\sqrt{\frac{C_2e^{\mu CT^{**}}}{2}}\left\|[u^{k+1}(0)-u^k(0)]\right\|_{H_{0,\delta}}+\sqrt{\frac{C_1e^{\mu CT^{**}}}{2}}|||[u^k-u^{k-1}]|||_{0,\delta,T^{**}}.
\end{aligned}$$

Thus, taking $T^{**}$ sufficiently small so that $\Lambda:=\sqrt{\frac{C_1e^{\mu CT^{**}}}{2}}<1$ and putting $\beta_k=\sqrt{\frac{C_2e^{\mu CT^{**}}}{2}}\left\|[u^{k+1}(0)-u^k(0)]\right\|_{H_{0,\delta}}$ completes the proof of the Lemma. ■

Lemma 4.12 implies that $\{u^k\}$ is a Cauchy sequence in $C([0,T^{**}],H_{0,\delta})$. Combing this with the intermediate estimates $\|u\|_{H_{s',\delta}} \le \|u\|_{H_{s,\delta}}^{\frac{s'}{s}} \|u\|_{H_{0,\delta}}^{1-\frac{s'}{s}}$ (see Proposition 6.3) and Lemma 4.9 (A), we conclude that $\{u^k\}$ is a Cauchy sequence in $C([0,T^{**}],H_{s',\delta})$ for any $s'<s$. Therefore there is a unique $u\in C([0,T^{**}],H_{s',\delta})$ such that

$$|||u^k-u|||_{s',\delta,T^{**}} \to 0 \qquad \text{for any} \qquad s'<s. \tag{4.59}$$

Taking $\frac{5}{2}<s'<s$ and utilizing the embedding Theorem 6.9, we have

$$u^k\to u \qquad \text{in} \qquad C\left([0,T^{**}],C^1_\beta(\mathbb{R}^3)\right) \qquad \text{for any} \quad \beta\le\delta+\frac{3}{2},$$

where $C^1_\beta(\mathbb{R}^3)$ is the class for which the norm

$$\sup_{\mathbb{R}^3}\left((1+|x|)^\beta|u(x)|+\sum_{a=1}^3(1+|x|)^{\beta+1}|\partial_a u(x)|\right)$$

is finite. From (4.38)

$$\partial_t u^{k+1}=\left(A^0(u^k;t,x)\right)^{-1}\left[\sum_{a=1}^3 A^a(u^k;t,x)\partial_a u^{k+1}+B(u^k;t,x)u^{k+1}+F(u^k;t,x)\right],$$

therefore by Corollary 6.8 and Proposition 6.3, we obtain $\partial_t u^k\to\partial_t u$ in $H_{s-1,\delta+1}$. Hence

$$\partial_t u^k\to\partial_t u \qquad \text{in} \qquad C\left([0,T^{**}],C_{\beta+1}(\mathbb{R}^3)\right) \qquad \text{for any} \quad \beta\le\delta+\frac{3}{2}.$$

Thus $u\in C^1\left(\mathbb{R}^3\times[0,T^{**}]\right)$ is a classical solution of the nonlinear system (4.1). Moreover, it follows from Lemma 4.9 (B) that $u\in \mathrm{Lip}\left([0,T^{**}],H_{s-1,\delta+1}\right)$. Our next task is to show that $u^k$ converges weakly to $u$ in $H_{s,\delta}$.

## 4.6 Weak Convergence

We first define the standard inner-product on $H_{s,\delta}$. For two vector valued functions $v,\phi\in H^s$, the expression

$$\langle v,\phi\rangle_s=\int\left(\Lambda^s v\right)^T\Lambda^s(\phi)dx=\int(1+|\xi|^2)^s\hat{v}^T\bar{\hat{\phi}}d\xi.$$

is an inner-product on $H^s$. Utilizing Definition 3.1 and Corollary 5.6 we see that

$$\langle v,\phi\rangle_{s,\delta}=\sum_j 2^{(\frac{3}{2}+\delta)2j}\left\langle(\psi_j^2 v)_{(2^j)},(\psi_j^2\phi)_{(2^j)}\right\rangle_s$$

is inner-product on $H_{s,\delta}$. This definition coincides with Definition 4.3 in the case where $A$ is the identity matrix.

**Lemma 4.13** ***(Weak Convergence)*** *For any $\phi\in H_{s,\delta}$, we have*

$$\lim_k\left\langle u^k(t),\phi\right\rangle_{s,\delta}=\langle u(t),\phi\rangle_{s,\delta} \tag{4.60}$$

*uniformly for $0\le t\le T^{**}$. Consequently*

$$\|u(t)\|_{H_{s,\delta}}\le\liminf_k\|u^k(t)\|_{H_{s,\delta}} \tag{4.61}$$

*and hence the solution $u$ of the initial value problem (4.1) belongs to $L^\infty\left([0,T^{**}],H_{s,\delta}\right)$.*

In order to show Lemma 4.13 we need the below property.

**Proposition 4.14** *Let $s < \frac{s'+s''}{2}$, $v \in H_{s',\delta}$, $\phi \in H_{s'',\delta}$. Then we have*

$$\left|\langle v,\phi\rangle_{s,\delta}\right| \leq \|v\|_{H_{s',\delta}}\|\phi\|_{H_{s'',\delta}}. \tag{4.62}$$

**Proof (of Proposition 4.14)** Elementary arguments show that

$$|\langle v,\phi\rangle_s| \leq \|v\|_{H^{s'}}\|\phi\|_{H^{s''}}.$$

Applying it term-wise and using the Cauchy-Schwarz inequality we have

$$\begin{aligned}
\left|\langle v,\phi\rangle_{s,\delta}\right| &\leq \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j}\left|\left\langle \left(\psi_j^2 v\right)_{2^j}, \left(\psi_j^2\phi\right)_{2^j}\right\rangle_s\right| \\
&\leq \sum_{j=0}^{\infty}\left(2^{\left(\frac{3}{2}+\delta\right)j}\left\|\left(\psi_j^2 v\right)_{2^j}\right\|_{H^{s'}}\right)\left(2^{\left(\frac{3}{2}+\delta\right)j}\left\|\left(\psi_j^2 \phi\right)_{2^j}\right\|_{H^{s''}}\right) \\
&\leq \left(\sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j}\left\|\left(\psi_j^2 v\right)_{2^j}\right\|_{H^{s'}}^2\right)^{\frac{1}{2}}\left(\sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j}\left\|\left(\psi_j^2 \phi\right)_{2^j}\right\|_{H^{s''}}^2\right)^{\frac{1}{2}} \\
&= \|v\|_{H_{s',\delta}}\|\phi\|_{H_{s'',\delta}}
\end{aligned}$$

■

**Proof (of Lemma 4.13)** Take $s'$ and $s''$ such that $s' < s < s''$ and $s < \frac{s'+s''}{2}$. For a given $\phi \in H_{s,\delta}$ and positive $\epsilon$, we may find by Theorem 6.10 (b), $\tilde{\phi} \in H_{s'',\delta}$ such that

$$\|\phi - \tilde{\phi}\|_{H_{s,\delta}} \leq \frac{\epsilon}{2R} \quad \text{and} \quad \|\tilde{\phi}\|_{H_{s'',\delta}} \leq C(\epsilon)\|\phi\|_{H_{s,\delta}}, \tag{4.63}$$

where $R$ is the positive number appearing in (4.36). Now,

$$\begin{aligned}
\left\langle u^k(t) - u(t),\phi\right\rangle_{s,\delta} &= \left\langle u^k(t)-u(t),\tilde{\phi}\right\rangle_{s,\delta} + \left\langle u^k(t)-u(t),\left(\phi-\tilde{\phi}\right)\right\rangle_{s,\delta} \\
&= I_k + II_k.
\end{aligned}$$

Therefore Proposition 4.14, (4.63) and (4.59) imply that

$$|I_k| \leq \|u^k(t)-u(t)\|_{H_{s',\delta}}\|\tilde{\phi}\|_{H_{s'',\delta}} \leq \|u^k(t)-u(t)\|_{H_{s',\delta}}C(\epsilon)\|\phi\|_{H_{s,\delta}} \to 0.$$

While in the second estimate we use Lemma 4.9 (A) and get

$$\begin{aligned}
|II_k| &\leq \|u^k(t)-u(t)\|_{H_{s,\delta}}\|\phi-\tilde{\phi}\|_{H_{s,\delta}} \\
&\leq \left(\|u^k(t)-u_0^0\|_{H_{s,\delta}} + \|u(t)-u_0^0\|_{H_{s,\delta}}\right)\|\phi-\tilde{\phi}\|_{H_{s,\delta}} \leq \frac{2R\epsilon}{2R} = \epsilon.
\end{aligned}$$

Thus,

$$\limsup_k \left|\left\langle u^k(t)-u(t),\phi\right\rangle_{s,\delta}\right| \leq \epsilon$$

which completes the proof of the limit (4.60). ■

For each $k$, $\left\langle u^k(t),\phi\right\rangle_{s,\delta}$ is continuous for $t \in [0,T^{**}]$ and by Lemma 4.13 it convergences uniformly to $\langle u(t),\phi\rangle_{s,\delta}$, hence $\langle u(t),\phi\rangle_{s,\delta}$ is a continuous function of $t$ for any $\phi \in H_{s,\delta}$ and we have obtained the following:

**Theorem 4.15** ***(Existence)*** *Under conditions (H1)-(H4) and (4.33) there is $u \in C^1\left(\mathbb{R}^3 \times [0,T^{**}]\right)$ a classical solution to the hyperbolic system (4.1) such that $u(t,x) \in \overline{G_2}$ and*

$$u \in L^\infty\left([0,T^{**}],H_{s,\delta}\right) \cap C_w\left([0,T^{**}],H_{s,\delta}\right) \cap \operatorname{Lip}\left([0,T^{**}],H_{s-1,\delta+1}\right), \tag{4.64}$$

*where $C_w$ means continuous in the weak topology of $H_{s,\delta}$.*

## 4.7 Well-posedness

In this subsection we well prove continuity in $H_{s,\delta}$-norm and uniqueness.

**Theorem 4.16** ***(Uniqueness)*** *Assume conditions (H1)-(H4) and (4.33) hold. If $u_1(t,x)$ and $u_2(t,x)$ are classical solutions to the hyperbolic system (4.1) such that $u_1, u_2 \in \overline{G_2}$, then $u_1 \equiv u_2$.*

**Proof (of Theorem 4.16)** Let $u_1$ and $u_2$ be a solutions to the hyperbolic system hyperbolic system (4.1) with the same initial data and let $V(t,x) = u_1(t,x) - u_2(t,x)$. Then $V$ satisfies the equation

$$A^0(u_1;t,x)\partial_t V = \sum_{a=1}^{3} A^a(u_1;,x)\partial_a V + B(u_1;t,x)V + G \tag{4.65}$$

with the initial condition $V(0,x) = 0$ and where

$$\begin{aligned} G = &\left[A^0(u_1;t,x) - A^0(u_2)\right]\partial_t u_1 + \sum_{a=1}^{3}[A^a(u_1;t,x) - A^a(u_2;t,x))]\partial_a u_1 \\ &+ [B(u_1;t,x) - B(u_2;t,x)]\, u_1 + [F(u_1;t,x) - F(u_2;t,x))]. \end{aligned}$$

Applying Proposition 4.10 to (4.65), we have

$$\frac{d}{dt}\langle V,V\rangle_{L^2_\delta,A^0(u_1)} \leq \mu C\langle V,V\rangle_{L^2_\delta,A^0(u_1)} + \|G\|^2_{L^2_\delta}.$$

Let $T \leq T^*$, then Gronwall's inequality and the equivalence (4.50) imply

$$|||V|||^2_{0,\delta,T} \leq C_1 e^{C\mu T}\int_0^T \|G(t)\|^2_{L^2_\delta}dt.$$

Similar estimation as done in (4.57) yield that $\|G(t)\|^2_{L^2_\delta} \leq C_2\|V(t)\|^2_{L^2_\delta}$. Hence,

$$|||V|||^2_{0,\delta,T} \leq C_3 e^{C\mu T}T \quad |||V|||^2_{0,\delta,T}. \tag{4.66}$$

Thus, if $T$ is sufficiently small, then (4.66) leads to a contradiction unless $V \equiv 0$. ∎

**Theorem 4.17** ***(Continuation in norm)*** *Under conditions (H1)-(H4) and (4.33), any solutions $u$ to the hyperbolic system (4.1) which satisfies $u(t,x) \in \overline{G_2}$ and the regularity condition (4.64), satisfies in addition*

$$u \in C\left([0,T^{**}], H_{s,\delta}\right) \cap C^1\left([0,T^{**}], H_{s-1,\delta+1}\right). \tag{4.67}$$

**Proof (of Theorem 4.17)** We first treat the continuity $C\left([0,T^{**}], H_{s,\delta}\right)$. Since $u$ is a solution of initial value problem (4.1) which is reversible in time, is sufficient to show that

$$\lim_{t\downarrow 0}\|u(t) - u(0)\|_{H_{s,\delta}} = \lim_{t\downarrow 0}\|u(t) - u_0\|_{H_{s,\delta}} = 0. \tag{4.68}$$

We shall use the following known argument: suppose $\{w_n\}$ is a sequence in Hilbert space which converge weakly to $w_0$ and $\limsup_n \|w_n\| \leq \|w_0\|$, then $\lim_n \|w_n - w_0\| = 0$. We are going to use the equivalence norm $\|\cdot\|_{H_{s,\delta,A^0(u(0))}}$, so we need to show

$$\limsup_{t\downarrow 0}\|u(t)\|_{H_{s,\delta,A^0(u(0))}} \leq \|u_0\|_{Hs,\delta,A^0(u(0))}. \tag{4.69}$$

Let $\{u^k(t)\}$ be the sequence which is defined by the iteration process (4.38). It follows from the uniqueness Theorem 4.16 and (4.61) that

$$\|u(t)\|_{H_{s,\delta,A^0(u(t))}} \leq \liminf_k \|u^k(t)\|_{Hs,\delta,A^0(u(t))}, \tag{4.70}$$

where the limit above is uniformly in $t$. Applying the energy estimate (4.30), we have

$$\frac{d}{dt}\|u^{k+1}(t)\|^2_{H_{s,\delta,A^0(u^k(t))}} \le C\left(\mu\|u^{k+1}(t)\|^2_{H_{s,\delta,A^0(u^k(t))}} + 1\right).$$

So Gronwall's inequality yields

$$\|u^{k+1}(t)\|^2_{H_{s,\delta,A^0(u^k(t))}} \le e^{C\mu t}\left[\|u^{k+1}(0)\|^2_{H_{s,\delta,A^0(u^k(0))}} + Ct\right]. \tag{4.71}$$

Take now arbitrary $\epsilon > 0$, since $u^k(t) \to u(t)$ uniformly in $[0, T^{**}]$, we see from the inner-product (4.7) that there is $k_0$ such that

$$\|v(t)\|_{H_{s,\delta,A^0(u(t))}} \le (1+\epsilon)\|v(t)\|_{H_{s,\delta,A^0(u^k(t))}} \qquad \text{for} \qquad k \ge k_0. \tag{4.72}$$

Combing (4.70), (4.71), (4.72) and (4.37) with the fact that $u^k(t) \to u(t)$ uniformly in $[0, T^{**}]$, we obtain

$$\begin{aligned}
\limsup_{t\downarrow 0}\|u(t)\|^2_{H_{s,\delta,A^0(0)}} &= \limsup_{t\downarrow 0}\|u(t)\|^2_{H_{s,\delta,A^0(u(t))}}\\
&\le \limsup_{t\downarrow 0}\left(\liminf_k \|u^{k+1}(t)\|^2_{H_{s,\delta,A^0(u(t))}}\right)\\
&\le \limsup_{t\downarrow 0}\left(\liminf_k (1+\epsilon)^2\|u^{k+1}(t)\|^2_{H_{s,\delta,A^0(u^k(t))}}\right)\\
&\le \limsup_{t\downarrow 0}\left(\liminf_k e^{C\mu t}\left[(1+\epsilon)^2\|u^{k+1}(0)\|^2_{H_{s,\delta,A^0(u^k(0))}} + Ct\right]\right)\\
&= \limsup_{t\downarrow 0}\left(e^{C\mu t}\left[(1+\epsilon)^2\|u_0\|^2_{H_{s,\delta,A^0(u(0))}} + Ct\right]\right)\\
&= (1+\epsilon)^2\|u_0\|^2_{H_{s,\delta,A^0(u(0))}}
\end{aligned}$$

which proves (4.69).

It remains to show that $\lim_{t\to t_0}\left(\|\partial_t u(t) - \partial_t u(t_0)\|_{H_{s-1,\delta+1}}\right) = 0$. Now,

$$\partial_t u = \left(A^0(u;t,x)\right)^{-1}\left\{\sum_{a=1}^{3} A^a(u;t,x)\partial_a u + B(u;t,x)u + F(u;t,x)\right\}. \tag{4.73}$$

By the first step of the proof, $\|\partial_a u(t) - \partial_a u(t_0)\|_{H_{s-1,\delta+1}} \to 0$ and $\|u(t) - u(t_0)\|_{H_{s,\delta}} \to 0$. At this stage we apply Corollary 6.8 to the right hand of (4.73) and this completes the proof of Theorem 4.17.

■

## 4.8 Local existence for the evolution equations of Einstein-Euler system

In the previous subsections we have established the well-posedness of the first order symmetric hyperbolic system in the $H_{s,\delta}$ spaces. We would like to apply this result to the coupled system (2.26) and (2.9).

The unknowns of the evolution equations are the gravitational field $g_{\alpha\beta}$ and its first order partial derivatives $\partial_\alpha g_{\gamma\delta}$, the Makino variable $w$ and the velocity vector $u^\alpha$. We represent them by the vector

$$U = \left(g_{\alpha\beta} - \eta_{\alpha\beta}, \partial_a g_{\gamma\delta}, \partial_0 g_{\gamma\delta}, w, u^a, u^0 - 1\right), \tag{4.74}$$

here $\eta_{\alpha\beta}$ denotes the Minkowski metric. The initial data for equation (2.26) are

$$g_{ab|_M} = h_{ab}, \quad g_{0b|_M} = 0, \quad g_{00|_M} = -1, \quad -\frac{1}{2}\partial_0 g_{ab|_M} = K_{ab}.$$

where $(h_{ab}, K_{ab})$ are given by (3.4), and (3.5) for equation (2.9). Hence $g_{ab|_M} - \eta_{ab} = h_{ab} - I \in H_{s,\delta}$ and $(w_{|_M}, u^a{}_{|_M}, u^0{}_{|_M} - 1) \in H_{s-1,\delta+2}$. Therefore we conclude that

$$U(0,\cdot) \in H_{s,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}. \tag{4.75}$$

In this situation we cannot apply directly Theorem 4.1. The idea to overcome this obstacle is the following. We first introduce some more convenience notations: $\mathbf{g} = g_{\alpha\beta} - \eta_{\alpha,\beta}$, $\partial\mathbf{g} = \partial_\alpha g_{\gamma\delta}$ (that is, $\partial\mathbf{g}$ is the set of all first order partial derivatives), $\mathbf{v} = (w, u^a, u^0 - 1)$ and $U = (\mathbf{g}, \partial\mathbf{g}, \mathbf{v})$. Since $H_{s,\delta} \subset H_{s-1,\delta}$, it follows from (4.75) that

$$U(0,\cdot) \in H_{s-1,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}. \tag{4.76}$$

If we prove the existence of $U(t,x)$ which is a solution to the coupled systems (2.26) and (2.9) with initial data in the form of (4.76) and such that $U(t,\cdot) \in H_{s-1,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$ and it is continuous with respect to this norm, then from inequality

$$\|\mathbf{g}\|_{H_{s,\delta}} \lesssim \left(\|\mathbf{g}\|_{H_{s-1,\delta}} + \|\partial\mathbf{g}\|_{H_{s-1,\delta+1}}\right), \tag{4.77}$$

we will get that $U(t,\cdot) \in H_{s,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$ and it will be continuous with respect to the norm of $H_{s,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$. Note that (4.77) certainly holds for the integral representation of the norm (5.3), and by Theorem 5.2 it holds also for the $H_{s,\delta}$ norm.

In order to achieve this we carefully examine the structure of the coupled systems (2.26) and (2.9). According to Conclusion 2.4, we can write Einstein-Euler system in the form:

$$A^0(U)\partial_t U = \sum_{a=1}^{3} A^a(U)\partial_a U + B(U)U, \tag{4.78}$$

where $A^\alpha$ and $B$ are $55 \times 55$ matrices such that

$$A^\alpha = \left(\begin{array}{c|c|c} I_{10} & \mathbf{0}_{10\times 40} & \mathbf{0}_{10\times 5} \\ \hline \mathbf{0}_{40\times 10} & \widetilde{A^\alpha}(\mathbf{g}) & \mathbf{0}_{40\times 5} \\ \hline \mathbf{0}_{5\times 10} & \mathbf{0}_{5\times 40} & \widehat{A^\alpha}(\mathbf{g}, \partial\mathbf{g}, \mathbf{v}) \end{array}\right) \tag{4.79}$$

and

$$B = \left(\begin{array}{c|c|c} \mathbf{0}_{10} & \mathbf{b}_{10\times 40} & \mathbf{0}_{10\times 5} \\ \hline & \widetilde{B}(\mathbf{g}, \partial\mathbf{g}, \mathbf{v}) & \\ \hline \mathbf{0}_{5\times 10} & \mathbf{0}_{5\times 40} & \mathbf{0}_{5\times 5} \end{array}\right). \tag{4.80}$$

Here $\widetilde{A^\alpha}(\mathbf{g})$ is $40 \times 40$ matrix which represents system (2.26), $\widehat{A^\alpha}(\mathbf{g}, \partial\mathbf{g}, \mathbf{v})$ is $5 \times 5$ matrix of system (2.9). Both of them are symmetric and both $\widetilde{A^0}(\mathbf{g})$ and $\widehat{A^0}(\mathbf{g}, \partial\mathbf{g}, \mathbf{v})$ are positive definite matrices; $\widetilde{B}(\mathbf{g}, \partial\mathbf{g}, \mathbf{v})$ is $40 \times 55$ matrix, and $\mathbf{b}_{10\times 40}$ is a constant matrix.

A natural norm of $U = (\mathbf{g}, \partial\mathbf{g}, \mathbf{v})$ on the product space $H_{s-1,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$ is

$$|||U|||^2_{H_{s-1,\delta}} = \|\mathbf{g}\|^2_{H_{s-1,\delta}} + \|\partial\mathbf{g}\|^2_{H_{s-1,\delta+1}} + \|\mathbf{v}\|^2_{H_{s-1,\delta+2}}. \tag{4.81}$$

Note that from Proposition (6.4) and Theorem 6.6 we have that $A^\alpha U, BU \in H_{s-1,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$, whenever $U \in H_{s-1,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$.

We formulate an inner-product in accordance with the norm (4.81) and the structure of $A^0$. Let $U_1 = (\mathbf{g}_1, \partial\mathbf{g}_1, \mathbf{v}_1)$ and $U_2 = (\mathbf{g}_2, \partial\mathbf{g}_2, \mathbf{v}_2)$, similarly to (4.7) we set

$$\begin{aligned}
&\langle U_1, U_2\rangle_{s-1,\delta,A^0} \\
:= &\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \int \left[\Lambda^{s-1}\left((\psi_j^2\mathbf{g}_1)_{(2^j)}\right)\right]^T \left[\Lambda^{s-1}\left((\psi_j^2\mathbf{g}_2)_{(2^j)}\right)\right] dx \\
+ &\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta+1)2j} \left[\Lambda^{s-1}\left((\psi_j^2\partial\mathbf{g}_1)_{(2^j)}\right)\right]^T (\widetilde{A}^0)_{(2^j)} \left[\Lambda^{s-1}\left((\psi_j^2\partial\mathbf{g}_2)_{(2^j)}\right)\right] dx \\
+ &\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta+2)2j} \left[\Lambda^{s-1}\left((\psi_j^2\mathbf{v}_1)_{(2^j)}\right)\right] (\widehat{A}^0)_{(2^j)} \left[\Lambda^{s-1}\left((\psi_j^2\mathbf{v}_2)_{(2^j)}\right)\right] dx
\end{aligned} \tag{4.82}$$

and $|||U|||^2_{H_{s-1,\delta,A^0}} = \langle U, U\rangle_{s-1,\delta,A^0}$. Since $A^0$ is positive definite, $|||U|||_{H_{s-1,\delta,A^0}} \sim |||U|||_{H_{s-1,\delta}}$

We can now repeat all the arguments and estimations of subsections 4.2-4.7, which are applied term-wise to the norm (4.81) and inner-product (4.82), and in this way we extend Theorem 4.1 to the product space:

**Theorem 4.18** ***(Well posedness of hyperbolic systems in product spaces)*** *Let $s-1 > \frac{5}{2}$, $\delta \geq -\frac{3}{2}$ and assume the coefficient of (4.78) are of the form (4.79) and (4.80). If $U_0 \in H_{s-1,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$ and satisfies*

$$\frac{1}{\mu} U_0^T U_0 \leq U_0^T A^0 U_0 \leq \mu U_0^T U_0, \qquad \mu \in \mathbb{R}^+ \tag{4.83}$$

*then there exits a positive $T$ which depends on $|||U_0|||_{H_{s-1,\delta}}$ and a unique $U(t,x)$ a solution to (4.78) such that $U(0,x) = U_0(x)$ and in addition it satisfies*

$$U \in C([0,T], H_{s-1,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}) \cap C^1([0,T], H_{s-2,\delta+1} \times H_{s-2,\delta+2} \times H_{s-2,\delta+3}). \tag{4.84}$$

**Corollary 4.19** ***(Solution to the gravitational field and the fluid)*** *Let $\frac{7}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$ and $\delta > -\frac{3}{2}$. Then there exists a positive $T$, a unique gravitational field $g_{\alpha\beta}$ solution to (2.26) and a unique $(w, u^\alpha)$ solution to Euler equation (2.9) such that*

$$g_{\alpha\beta} - \eta_{\alpha\beta} \in C([0,T], H_{s,\delta}) \cap C^1([0,T], H_{s-1,\delta+1}) \tag{4.85}$$

*and*

$$(w, u^a, u^0 - 1) \in C([0,T], H_{s-1,\delta+2}) \cap C^1([0,T], H_{s-2,\delta+3}). \tag{4.86}$$

**Proof (of Corollary 4.19)** Theorem 3.4 implies that the initial data for $g_{\alpha\beta}$ belong to $H_{s,\delta}$ and initial data for $(w, u^\alpha)$ are in $H_{s-1,\delta+2}$. Thus $U(0,\cdot) \in H_{s-1,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$, where $U$ is given by (4.74). In addition, the continuity of $A^0$ implies that the vector $U(0,\cdot)$ satisfies (4.83). Therefore Theorem 4.18 with inequality (4.77) give the desired result. ■

**Acknowledgement:** The problem we worked out goes back to Alan Rendall and we would like to thank him for enlightening discussions. The second author would like to thank Victor Ostrovsky for many valuable conversations.

# Appendix

## 5 The construction of the Spaces $H_{s,\delta}$:

The weighted Sobolev spaces of integer order below were introduced by Cantor [6] and independently by Nirenberg and Walker [21]. Nirenberg and Walker initiate the study of elliptic operators in these spaces, while Cantor used them to solve the constraint equations on asymptotically flat manifolds. For an nonnegative integer $m$ and a real $\delta$ we define a norm

$$\left(\|u\|^{*}_{m,\delta}\right)^2 = \sum_{|\alpha|\leq m}\int \left(\langle x\rangle^{\delta+|\alpha|}|\partial^\alpha u|\right)^2 dx, \tag{5.1}$$

where $\langle x\rangle = 1+|x|$. The space $H_{s,m}$ is the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm (5.1). Note that the weight varies with the derivatives.

Here we will repeat Triebel's extension of these spaces into a fractional order, [27],[28]. Let $s = m+\lambda$, where $m$ is a nonnegative integer and $0<\lambda<1$. One possibility of extending the ordinary integer order Sobolev spaces is the *Lipschitz-Sobolevskij Spaces*, having a norm

$$\|u\|^2_{m+\lambda,2} = \sum_{|\alpha|\leq m}\int |\partial^\alpha u|^2 dx + \sum_{|\alpha|=m}\int\int \frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|^2}{|x-y|^{3+\lambda 2}}dxdy. \tag{5.2}$$

Hence, a reasonable definition of *weighted fractional Sobolev norm* is a combination of the norm (5.1) with (5.2):

$$\left(\|u\|^{*}_{s,\delta}\right)^2 = \begin{cases} \displaystyle\sum_{|\alpha|\leq m}\int |\langle x\rangle^{\delta+|\alpha|}\partial^\alpha u|^2 dx, & s=m \\ \left.\begin{array}{l}\displaystyle\sum_{|\alpha|\leq m}\int |\langle x\rangle^{\delta+|\alpha|}\partial^\alpha u|^2 dx \\ \displaystyle+\sum_{|\alpha|=m}\int\int \frac{|\langle x\rangle^{m+\lambda+\delta}\partial^\alpha u(x)-\langle y\rangle^{m+\lambda+\delta}\partial^\alpha u(y)|^2}{|x-y|^{3+2\lambda}}dxdy\end{array}\right\}, & s=m+\lambda. \end{cases} \tag{5.3}$$

here $m$ is a nonnegative integer and $0<\lambda<1$. The space $H_{s,\delta}$ is the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm (5.3).

The norm (5.3) is essential for the understating of the connections between the integer and the fractional order. But it has a disadvantage, namely, the double integral makes it almost impossible to establish any property (embedding, a priori estimate, etc.) needed for PDEs. We are therefore looking for an equivalent definition of the norm (5.3).

Let $K_j = \{x : 2^{j-3}\leq |x|\leq 2^{j+2}\}$, $(j=1,2,...)$ and $K_0=\{x:|x|\leq 4\}$. Let $\{\psi_j\}_{j=0}^\infty\subset C_0^\infty(\mathbb{R}^3)$ be a sequence such that $\psi_j(x)=1$ on $K_j$, $\mathrm{supp}(\psi_j)\subset\{x: 2^{j-4}\leq|x|\leq 2^{j+3}\}$, for $j\geq 1$, $\mathrm{supp}(\psi_0)\subset\{x:|x|\leq 2^3\}$ and

$$|\partial^\alpha\psi_j(x)|\leq C_\alpha 2^{-|\alpha|j}, \tag{5.4}$$

where the constant $C_\alpha$ does not depend on $j$.

We define now,

$$\left(\|u\|^{\bigstar}_{s,\delta}\right)^2 = \begin{cases} \displaystyle\sum_{j=0}^\infty\left(2^{\delta 2j}\|\psi_j u\|^2_{L^2} + 2^{(\delta+m)2j}\sum_{|\alpha|=m}\|\partial^\alpha(\psi_j u)\|^2_{L^2}\right), & s=m \\ \left.\begin{array}{l}\displaystyle\sum_{j=0}^\infty\left(2^{\delta 2j}\|\psi_j u\|^2_{L^2} + 2^{(\delta+m)2j}\sum_{|\alpha|=m}\|\partial^\alpha(\psi_j u)\|^2_{L^2}\right) \\ +\sum_{j=0}^\infty 2^{(\delta+m+\lambda)2j}\left(\sum_{|\alpha|=m}\int\int\frac{|\partial^\alpha(\psi_j u)(x)-\partial^\alpha(\psi_j u)(y)|^2}{|x-y|^{3+2\lambda}}dxdy\right),\end{array}\right\} & s=m+\lambda. \end{cases} \tag{5.5}$$

**Proposition 5.1** ***(Equivalence of norms)*** *There are two positive constants* $c_0$ *and* $c_1$ *depending only on* $s$, $\delta$ *and the constants in (5.4) such that*

$$c_0\|u\|^{\bigstar}_{s,\delta}\leq\|u\|^{*}_{s,\delta}\leq c_1\|u\|^{\bigstar}_{s,\delta}. \tag{5.6}$$

This equivalence was proved in [27] (see also [2]).

We express these norms in terms of Fourier transform. Let

$$\hat{u}(\xi) = \mathcal{F}(u)(\xi) = \frac{1}{(2\pi)^3}\int u(x)e^{-ix\cdot\xi}dx$$

denotes the Fourier transform, put

$$\Lambda^s u = \mathcal{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}u), \tag{5.7}$$

and let $H^s$ denotes the Bessel Potentials space having the norm

$$\|u\|_{H^s}^2 = \|\Lambda^s u\|_{L^2}^2 = \int (1+|\xi|^2)^s|\hat{u}(\xi)|^2 d\xi. \tag{5.8}$$

We also set

$$\|u\|_{h^s}^2 = \|\mathcal{F}^{-1}(|\xi|^s\mathcal{F}u)\|_{L^2}^2 = \int (|\xi|^s|\hat{u}(\xi)|)^2 d\xi.$$

It is well known that (see e. g. [12]; p. 240-241)

$$\|u\|_{h^s}^2 \simeq \begin{cases} \sum_{|\alpha|=m}\int |\partial^\alpha u|^2 dx & s=m \\ \sum_{|\alpha|=m}\int\int \frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|^2}{|x-y|^{3+2\lambda}}dx & s=m+\lambda \end{cases} \tag{5.9}$$

and since $(1+|\xi|^2)^s \simeq (1+|\xi|^s)$,

$$\|u\|_{H^s}^2 \simeq \left(\|u\|_{L^2}^2 + \|u\|_{h^s}^2\right). \tag{5.10}$$

Hence, by (5.5),

$$\left(\|u\|_{s,\delta}^{\bigstar}\right)^2 \simeq \sum_{j=0}^{\infty}\left(2^{\delta 2j}\|\psi_j u\|_{L^2}^2 + 2^{(\delta+s)2j}\|\psi_j u\|_{h^s}^2\right) \tag{5.11}$$

We invoke now the scaling $u_\epsilon(x) := u(\epsilon x)$ $(\epsilon > 0)$, then simple calculations yields $\|u_\epsilon\|_{L^2}^2 = \epsilon^{-3}\|u\|_{L^2}^2$ and $\|u_\epsilon\|_{h^s}^2 = \epsilon^{2s-3}\|u\|_{h^s}^2$. Combining the later one with (5.10), we have

$$\|u_\epsilon\|_{H^s}^2 \simeq \epsilon^{-3}\left(\|u\|_{L^2} + \epsilon^{2s}\|u\|_{h^s}^2\right). \tag{5.12}$$

Setting $\epsilon = 2^j$, multiplying (5.12) by $2^{3j}$ and inserting it in (5.11), we conclude

$$\left(\|u\|_{s,\delta}^{\bigstar}\right)^2 \simeq \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\|(\psi_j u)_{2^j}\|_{H^s}^2. \tag{5.13}$$

The last one is the most convenience form of norm for applications and therefore the right hand side of (5.13) defines the norm of $H_{s,\delta}$ space (see Definition 3.1).

Combining Proposition 5.1 with (5.11) and (5.13) we get:

**Theorem 5.2** ***(Equivalence of norms, Triebel)*** *There are two positive constant $c_0$ and $c_1$ depending only on $s$, $\delta$ and the constants in (5.4) such that*

$$c_0\|u\|_{Hs,\delta} \leq \|u\|_{s,\delta}^* \leq c_1\|u\|_{Hs,\delta}. \tag{5.14}$$

**Remark 5.3** *Theorem 5.2 enables us to use both sorts of the norms (5.3) and (3.1), and for each application we will use the suitable type of norm.*

**Remark 5.4** *Let $s' \leq s$ and $\delta' \leq \delta$, then the inclusion $H_{s,\delta} \hookrightarrow H_{s',\delta'}$ follows easily from the representations (5.8) and (3.1) of the norms.*

**Remark 5.5** *The functions $\{\psi_j\}$ are constructed by means of a composition of exponential functions. Hence, for any positive $\gamma$ there holds*

$$c_1(\gamma,\alpha)|\partial^\alpha \psi_j^\gamma(x)| \le |\partial^\alpha \psi_j(x)| \le c_2(\gamma,\alpha)|\partial^\alpha \psi_j^\gamma(x)|. \tag{5.15}$$

*Therefore the equivalence (5.6) remains valid with $\psi_j^\gamma$ replacing $\psi_j$ and hence*

$$\sum_j 2^{(\frac{3}{2}+\delta)2j}\|(\psi_j^\gamma u)_{(2^j)}\|^2_{H^s} \simeq \left(\|u\|^\star_{s,\delta}\right)^2 \simeq \sum_j 2^{(\frac{3}{2}+\delta)2j}\|(\psi_j u)_{(2^j)}\|^2_{H^s}. \tag{5.16}$$

**Corollary 5.6** ***(Equivalence of norms)*** *For any positive $\gamma$, there are two positive constants $c_0$ and $c_1$ depending on $s$, $\delta$ and $\gamma$ such that*

$$c_0\|u\|^2_{H_{s,\delta}} \le \sum_j 2^{(\frac{3}{2}+\delta)2j}\|(\psi_j^\gamma u)_{(2^j)}\|^2_{H^s} \le c_1\|u\|^2_{H_{s,\delta}}. \tag{5.17}$$

# 6 Some Properties of $H_{s,\delta}$

**Theorem 6.1** ***(Complex interpolation, Triebel)***
*Let $0<\theta<1$, $0\le s_0<s_1$ and $s_\theta=\theta s_0+(1-\theta)s_1$, then*

$$[H_{s_0,\delta}, H_{s_1,\delta}]_\theta = H_{s_\theta,\delta}, \tag{6.1}$$

*where (6.1) is a complex interpolation.*

As a consequence of the interpolation Theorem 6.1 we get

**Corollary 6.2** ***($H_{s,\delta}$-norm of a derivative)***

$$\|\partial_i u\|_{H_{s-1,\delta+1}} \le \|u\|_{H_{s,\delta}} \tag{6.2}$$

**Proof (of Corollary 6.2)** Let $m$ be a positive integer and define $T: H_{m,\delta}\to H_{m-1,\delta+1}$ by $T(u)=\partial_i u$. Using the norm (5.1) we see that $\|T(u)\|_{H_{m-1,\delta+1}} \le \|u\|_{H_{m,\delta}}$. So (6.2) follows from Theorem 6.1. ∎

**Proposition 6.3** ***(An intermediate estimate)*** *Let $0<s'<s$, then*

$$\|u\|_{H_{s',\delta}} \le \|u\|^{\frac{s'}{s}}_{H_{s,\delta}}\|u\|^{1-\frac{s'}{s}}_{H_{0,\delta}}. \tag{6.3}$$

**Proof (of Proposition 6.3)** Using Hölder inequality we get $\|u\|_{H^{s'}} \le \|u\|^{\frac{s'}{s}}_{H^s}\|u\|^{1-\frac{s'}{s}}_{L^2}$ and applying it and using again Hölder inequality yields

$$\begin{aligned}
\|u\|^2_{H_{s',\delta}} &= \sum_j 2^{\left(\frac{3}{2}+\delta\right)2j}\left\|(\psi_j u)_{2^j}\right\|^2_{H^{s'}}\\
&\le \sum_j 2^{\left(\frac{3}{2}+\delta\right)2j\left(\frac{s'}{s}\right)}\left\|(\psi_j u)_{2^j}\right\|^{2\frac{s'}{s}}_{H^s}\quad 2^{\left(\frac{3}{2}+\delta\right)2j\left(\frac{s-s'}{s}\right)}\left\|(\psi_j u)_{2^j}\right\|^{2\frac{s-s'}{s}}_{L_2}\\
&\le \left(\sum_j 2^{\left(\frac{3}{2}+\delta\right)2j}\left\|(\psi_j u)_{2^j}\right\|^2_{H^s}\right)^{\frac{s'}{s}}\quad \left(\sum_j 2^{\left(\frac{3}{2}+\delta\right)2j}\|(\psi_j u)\|^2_{L_2}\right)^{\frac{s-s'}{s}}\\
&= \left(\|u\|_{H_{s,\delta}}\right)^{\frac{2s'}{s}}\quad \left(\|u\|_{H_{0,\delta}}\right)^{\frac{2(s'-1)}{s}}.
\end{aligned}$$

∎

## 6.1 Algebra

**Proposition 6.4** ***(Algebra in $H_{s,\delta}$)***
*If $s_1, s_2 \geq s$, $s_1 + s_2 > s + \frac{3}{2}$ and $\delta_1 + \delta_2 \geq \delta - \frac{3}{2}$, then*

$$\|uv\|_{H_{s,\delta}} \leq C\|u\|_{H_{s_1,\delta_1}}\|v\|_{H_{s_2,\delta_2}}. \tag{6.4}$$

**Proof (of Proposition 6.4)** By Corollary 5.6,

$$\|uv\|^2_{H_{s,\delta}} \simeq \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|\left(\psi_j^2 uv\right)_{2^j}\right\|^2_{H^s}. \tag{6.5}$$

We apply the classic algebra property $\|uv\|_{H^s} \leq C\|u\|_{H^{s_1}}\|v\|_{H^{s_2}}$ (see e. g. [26] Ch. 3, Section 5), to each term of the norm (6.5) and then we use Cauchy Schwarz inequality,

$$\begin{aligned}
&\|uv\|^2_{H_{s,\delta}} \leq C\sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|\left(\psi_j^2 uv\right)_{2^j}\right\|^2_{H^s} \\
&\leq C^2 \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|\left(\psi_j u\right)_{2^j}\right\|^2_{H^{s_1}} \left\|\left(\psi_j v\right)_{2^j}\right\|^2_{H^{s_2}} \\
&\leq C^2 \sum_{j=0}^{\infty} \left(2^{\left(\frac{3}{2}+\delta_1\right)2j} \left\|\left(\psi_j u\right)_{2^j}\right\|^2_{H^{s_1}}\right) \left(2^{\left(\frac{3}{2}+\delta_2\right)2j} \left\|\left(\psi_j v\right)_{2^j}\right\|^2_{H^{s_2}}\right) \\
&\leq C^2 \left(\sum_{j=0}^{\infty} \left(2^{\left(\frac{3}{2}+\delta_1\right)2j} \left\|\left(\psi_j u\right)_{2^j}\right\|^2_{H^{s_1}}\right)^2\right)^{\frac{1}{2}} \left(\sum_{j=0}^{\infty} \left(2^{\left(\frac{3}{2}+\delta_2\right)2j} \left\|\left(\psi_j v\right)_{2^j}\right\|^2_{H^{s_2}}\right)^2\right)^{\frac{1}{2}} \\
&\leq C^2 \left(\sum_{j=0}^{\infty} \left(2^{\left(\frac{3}{2}+\delta_1\right)2j} \left\|\left(\psi_j u\right)_{2^j}\right\|^2_{H^{s_1}}\right)\right) \left(\sum_{j=0}^{\infty} \left(2^{\left(\frac{3}{2}+\delta_2\right)2j} \left\|\left(\psi_j v\right)_{2^j}\right\|^2_{H^{s_2}}\right)\right) \\
&\leq C^2 \|u\|^2_{H_{s_1,\delta_1}} \|v\|^2_{H_{s_2,\delta_2}}.
\end{aligned}$$

■

## 6.2 Moser type estimates

Y. Meyer proved the below Moser type estimate [20].See also Taylor [26].

**Theorem 6.5** ***(Third Moser inequality for Bessel potentials spaces)***
*Let $F : \mathbb{R}^m \to \mathbb{R}^l$ be $C^{N+1}$ function such that $F(0) = 0$. Let $s > 0$ and $u \in H^s \cap L^\infty$. Then*

$$\|F(u)\|_{H^s} \leq K\|u\|_{H^s}, \tag{6.6}$$

*where*

$$K = K_N(F, \|u\|_{L^\infty}) \leq C\|F\|_{C^{N+1}}\left(1 + \|u\|^N_{L^\infty}\right), \tag{6.7}$$

*here $N$ is a positive integer such that $N \geq [s] + 1$.*

We generalize this important inequality to the $H_{s,\delta}$ spaces.

**Theorem 6.6** ***(Third Moser inequality in $H_{s,\delta}$)***
*Let $F : \mathbb{R}^m \to \mathbb{R}^l$ be $C^{N+1}$ function such that $F(0) = 0$. Let $s > 0$, $\delta \in \mathbb{R}$ and $u \in H_{s,\delta} \cap L^\infty$. Then*

$$\|F(u)\|_{H_{s,\delta}} \leq K\|u\|_{H_{s,\delta}}, \tag{6.8}$$

*The constant $K$ in (6.8) depends on one in (6.7) and in addition on $\delta$.*

**Proof (of Theorem 6.6)** Let $\{\psi_j\}$ be the sequence satisfying (5.4) and $\Psi_j(x) = \frac{1}{\varphi(x)}\psi_j(x)$, where $\varphi(x) = \sum_{j=0}^{\infty}\psi_j(x)$. From the properties of the sequence $\{\psi_j\}$, it follows that $1 \leq \varphi(x) \leq 7$. So the sequence $\{\Psi_j\} \subset C_0^{\infty}(\mathbb{R}^3)$ and $\sum_{j=0}^{\infty}\Psi_j(x) = 1$. From (5.12) we conclude that

$$\|u_\epsilon\|_{H^s}^2 \leq C\max\{\epsilon^{2s-3}, \epsilon^{-3}\}\|u\|_{H^s}^2 \tag{6.9}$$

and with the combination of (4.17) and Meyer's Theorem6.5 we have,

$$\begin{aligned}
\|F(u)\|_{H_{s,\delta}}^2 &= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\|(\psi_j(F(u))_{(2^j)}\|_{H^s}^2 \\
&= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\left\|\left(\psi_j F\left(\sum_{k=0}^{\infty}\Psi_k(x)u\right)\right)_{(2^j)}\right\|_{H^s}^2 \\
&= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\left\|\left(\psi_j F\left(\sum_{k=j-4}^{j+3}\Psi_k(x)u\right)\right)_{(2^j)}\right\|_{H^s}^2 \\
&\leq CK^2\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\sum_{k=j-4}^{j+3}\|\,(\Psi_k u)_{(2^j)}\,\|_{H^s}^2 \\
&\leq CK^2\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\sum_{k=j-4}^{j+3}\|\,((\Psi_k u)_{2^{j-k}})_{(2^k)}\,\|_{H^s}^2 \\
&\leq CK^2\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\sum_{k=j-4}^{j+3}\max\{2^{(2s-3)(j-k)}, 2^{-3(j-k)}\}\|\,(\Psi_k u)_{(2^k)}\,\|_{H^s}^2 \\
&\leq C(s)K^2\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\sum_{k=j-4}^{j+3}\|\,(\psi_k u)_{(2^k)}\,\|_{H^s}^2 \\
&\leq C(s,\delta)K^2\sum_{j=0}^{\infty}\sum_{k=j-4}^{j+3} 2^{(\frac{3}{2}+\delta)2k}\|\,(\psi_k u)_{(2^k)}\,\|_{H^s}^2 \\
&\leq 7C(s,\delta)K^2\sum_{k=0}^{\infty} 2^{(\frac{3}{2}+\delta)2k}\|\,(\psi_k u)_{(2^k)}\,\|_{H^s}^2 \leq 7C(s,\delta)K^2\|u\|_{H_{s,\delta}}^2.
\end{aligned} \tag{6.10}$$

■

**Remark 6.7** *If $F(0) \neq 0$ and $F(0) \in H_{s,\delta}$, then we can apply Theorem 6.6 to $\widetilde{F}(u) := F(u) - F(0)$ and get*

$$\|F(u)\|_{H_{s,\delta}} \leq \|\widetilde{F}(u)\|_{H_{s,\delta}} + \|F(0)\|_{H_{s,\delta}} \leq K\|u\|_{H_{s,\delta}} + \|F(0)\|_{H_{s,\delta}}. \tag{6.11}$$

We may apply Theorem 6.6 to the estimate the difference $F(u) - F(v)$.

**Corollary 6.8** ***(A difference estimate in $H_{s,\delta}$)*** *Suppose $F$ is a $C^{N+2}$ function and $u, v \in H_{s,\delta} \cap L^{\infty}$. Then*

$$\|F(u) - F(v)\|_{H_{s,\delta}} \leq C(\|u\|_{L^\infty}, \|v\|_{L^\infty})\left(\|u\|_{H_{s,\delta}} + \|v\|_{H_{s,\delta}}\right)\|u - v\|_{H_{s,\delta}}. \tag{6.12}$$

**Proof (of Corollary 6.8)** Put $\tilde{F}(u) = F(u) - F(0) - DF'(0)u$, then it suffices to show inequality (6.12) for $\tilde{F}$. Now,

$$\tilde{F}(u) - \tilde{F}(v) = \int_0^1 \left(D\tilde{F}\left(tu + (1-t)v\right)\right)(u - v)dt = G(u, v)(u - v), \tag{6.13}$$

where $G(u,v) = \int_0^1 D\tilde{F}\,(tu+(1-t)v)\,dt$. Since $G(0,0) = \int_0^1 D\tilde{F}(0)dt = 0$, we can apply Theorem 6.6 to $G(u,v)$ and get:

$$\|G(u,v)\|_{H_{s,\delta}} \leq C(\|u\|_{L^\infty}, \|v\|_{L^\infty})\left(\|u\|_{H_{s,\delta}} + \|v\|_{H_{s,\delta}}\right). \tag{6.14}$$

Applying Proposition (6.4) to the right side of (6.13), we have

$$\left\|\tilde{F}(u) - \tilde{F}(v)\right\|_{H_{s,\delta}} \leq C\,\|G(u,v)\|_{H_{s,\delta}}\,\|(u-v)\|_{H_{s,\delta}} \tag{6.15}$$

and its combination with (6.14) gives (6.12). ■

## 6.3 Embedding into the continuous

We introduce the following notations. For a nonnegative integer $m$ and $\beta \in \mathbb{R}$, we set

$$\|u\|_{C^m_\beta} = \sum_{|\alpha|\leq m} \sup_x \left((1+|x|)^{\beta+|\alpha|}|\partial^\alpha u(x)|\right)$$

Let $C^m_\beta$ be the functions space corresponding to the above norms.

**Theorem 6.9** ***(Embedding into the continuous)***
*If $s > \frac{3}{2}+m$ and $\delta + \frac{3}{2} \geq \beta$, then any $u \in H_{s,\delta}$ has a representative $\tilde{u} \in C^m_\beta$ satisfying*

$$\|\tilde{u}\|_{C^m_\beta} \leq C\|u\|_{H_{s,\delta}}. \tag{6.16}$$

**Proof (of Theorem 6.9)** We first show (6.16) when $m=0$. In order to make notations simpler we will use the convention $2^k = 0$ if $k<0$. Recall that $\psi_j(x) = 1$ on $K_j := \{2^{j-3} \leq |x| \leq 2^{j+2}\}$. Using the known embedding $\sup_x |u(x)| \leq C\|u\|_{H^s}$ (see e. g. [17]),we have

$$\begin{aligned}
&\sup_x (1+|x|)^\beta |u(x)| \leq 2^\beta \sup_{j\geq -1}\left(2^{\beta j} \sup_{\{2^j\leq|x|\leq 2^{j+1}\}} |u(x)|\right) \\
\leq& 2^\beta \sup_{j\geq -1}\left(2^{\beta j}\sup|\psi_j(x)u(x)|\right) = 2^\beta \sup_{j\geq -1}\left(2^{\beta j}\sup|\psi_j(2^j x)u(2^j x)|\right) \\
\leq& 2^\beta C \sup_{j\geq -1}\left(2^{\beta j}\|(\psi_j u)_{2^j}\|_{H^s}\right) \leq 2^\beta C \sup_{j\geq -1}\left(2^{(\frac{3}{2}+\delta)j}\|(\psi_j u)_{2^j}\|_{H^s}\right) \leq 2^\beta C\|u\|_{H_{s,\delta}}.
\end{aligned} \tag{6.17}$$

If $m>1$, $s > \frac{3}{2}+m$ and $\delta + \frac{3}{2} \geq \beta$, then $\partial^\alpha u \in H_{s-|\alpha|,\delta+|\alpha|}$ for $1 \leq |\alpha| \leq m$. So we may apply (6.17) to $\partial^\alpha u$ and obtain $\|\partial^\alpha u\|_{C_{\beta+k}} \leq C\|\partial^\alpha u\|_{H_{s-|\alpha|,\delta+|\alpha|}}$. ■

## 6.4 Density

**Theorem 6.10** ***(Density of $C_0^\infty$ functions)***

(a) *The class $C_0^\infty(\mathbb{R}^3)$ is dense in $H_{s,\delta}$.*

(b) *Given $u \in H_{s,\delta}$ and $s' > s \geq 0$. Then for $\rho > 0$ there is $u_\rho \in C_0^\infty(\mathbb{R}^3)$ and a positive constant $C(\rho)$ such that*

$$\|u_\rho - u\|_{H_{s,\delta}} \leq \rho \quad \textit{and} \quad \|u_\rho\|_{H_{s',\delta}} \leq C(\rho)\|u\|_{H_{s,\delta}}. \tag{6.18}$$

Property (a) was proved by Triebel [28]. We prove both of them here since (b) relies on (a).

**Proof (of Theorem 6.10)** Let $J_\epsilon$ be the standard mollifier, that is, $\text{supp}(J_\epsilon) \subset B(0,\epsilon)$, $\hat{J}_\epsilon(\xi) = \hat{J}_1(\epsilon\xi) = \hat{J}(\epsilon\xi)$ and $\hat{J}(0) = 1$. It is well known that for any $v \in H^s$, $\|J_\epsilon * v - v\|_{H^s} \to 0$ and that $J_\epsilon * v$ belongs to $C^\infty(\mathbb{R}^3)$. In addition, we claim that there is $C = C(\epsilon, s, s')$ such that

$$\|J_\epsilon * v\|_{H^{s'}} \leq C\|v\|_{H^s}. \tag{6.19}$$

Indeed, since $J \in C_0^\infty(\mathbb{R}^3)$, $|\hat{J}(\xi)| \leq C_m(1+|\xi|)^{-m}$ for any integer $m$. Therefore, for a given $s'$ and $\epsilon$, we chose $m$ and the constant $C(\epsilon,s,s')$ so that $(1+|\xi|^2)^{s'-s}|\hat{J}(\epsilon\xi)|^2 \leq C^2(\epsilon,s,s')$. Hence

$$\begin{aligned}\|J_\epsilon * v\|_{H^{s'}}^2 &= \int (1+|\xi|^2)^{s'}|\hat{J}(\epsilon\xi)|^2|\hat{v}(\xi)|^2 d\xi = \int (1+|\xi|^2)^{s}|\hat{v}(\xi)|^2(1+|\xi|^2)^{s'-s}\hat{J}(\epsilon\xi)|^2 d\xi \\ &\leq C^2(\epsilon,s,s')\int (1+|\xi|^2)^{s}|\hat{v}(\xi)|^2 d\xi = C^2(\epsilon,s,s')\|v\|_{H^s}^2.\end{aligned}$$

(a) Given $u \in H_{s,\delta}$ and $\rho > 0$ we may chose $N$ such that

$$\sum_{j=N-2}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\|(\psi_j(u)_{(2^j)}\|_{H^s}^2 \leq \rho^2.$$

Set now $u_N = \sum_{j=0}^{N}\Psi_k u$, where $\Psi_k$ is defined as in the proof of Theorem 6.6. We use and get

$$\|fu\|_{H^s} \leq C_s K\|u\|_{H^s} \tag{6.20}$$

$$\begin{aligned}\|u-u_N\|_{H_{s,\delta}}^2 &\leq \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\left\|\left(\psi_j\left(\sum_{k=N+1}^{\infty}\Psi_k u\right)\right)_{(2^j)}\right\|_{H^s}^2 \\ &= \sum_{j=N-2}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\left\|\left(\sum_{k=j-3}^{j+4}\psi_j\Psi_k u\right)_{(2^j)}\right\|_{H^s}^2 \leq C\sum_{j=N-2}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\sum_{k=j-3}^{j+4}\left\|(\psi_j u)_{(2^j)}\right\|_{H^s}^2 \\ &\leq 7C\sum_{j=N-2}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\left\|(\psi_j u)_{(2^j)}\right\|_{H^s}^2 = 7C\rho^2.\end{aligned}$$

Now $u_N$ has compact support, therefore $J_\epsilon * u_N \in C_0^\infty(\mathbb{R}^3)$ and

$$\|J_\epsilon * u_N - u_N\|_{H_{s,\delta}}^2 \leq \sum_{j=0}^{N+4} 2^{(\frac{3}{2}+\delta)2j}\left\|(\psi_j(J_\epsilon * u_N - u_N))_{(2^j)}\right\|_{H^s}^2 \to 0 \quad \text{as} \quad \epsilon \to 0.$$

(b) Let $u \in H_{s,\delta}$ and $\rho > 0$, then by (a) we can chose $N$ sufficiently large and $\epsilon$ small so that $\|J_\epsilon * u_N - u\|_{H_{s,\delta}} < \rho$ and by (6.19)

$$\begin{aligned}\|J_\epsilon * u_N\|_{H_{s',\delta}}^2 &\leq \sum_{j=0}^{N+4} 2^{(\frac{3}{2}+\delta)2j}\left\|(\psi_j(J_\epsilon * u_N))_{(2^j)}\right\|_{H^{s'}}^2 \\ &\leq C^2(\epsilon,s,s')\sum_{j=0}^{N+4} 2^{(\frac{3}{2}+\delta)2j}\left\|(\psi_j u_N))_{(2^j)}\right\|_{H^s}^2 \leq C^2C^2(\epsilon,s,s')\|u\|_{H_{s,\delta}}^2.\end{aligned}$$

Thus, $u_\rho = J_\epsilon * u_N$.

■

## References


[1] R. Bartnik and J. Isenberg. The constraint equations. In *The Einstein equations and the large scale behavior of gravitational fields*, pages 1–38. Birkhäuser, Basel, 2004.

[2] U. Brauer and L. Karp. Weighted spaces. Galilee Research Center for Applied Mathematics of ORT Braude College; http://brd90.ort.org.il/mathcentre/Preprints.aspx, 2006.

[3] U. Brauer and L. Karp. Local existence of classical solutions of the einstein–euler system using weighted sobolev spaces of fractional order. *Les Comptes rendus de l'Académie des sciences /Série Mathematique*, 345(1):49–54, 2007.

[4] Uwe Brauer and Lavi Karp. Well-posedness of the Einstein-Euler system in asymptotically flat spacetimes. arXiv:0810.5045, 2008.

[5] Uwe Brauer and Lavi Karp. Well-posedness of the Einstein-Euler System in asymptotically flat spacetimes: the constraint equations. Preprint, 2009.

[6] M. Cantor. Spaces of functions with asymptotic conditions on $\mathbb{R}^n$. *Indiana University Mathematics Journal*, 24(9):897–902, April 1975.

[7] Y. Choquet-Bruhat. Théorème d'existence pour certains systèmes d'équations aux dérivées partelles non linéaires. *Acta Math*, 88:141–225, 1952.

[8] Y. Choquet-Bruhat and J.W. York. The Cauchy Problem. In Alan Held, editor, *General relativity and Gravitation*, pages 99–172, New York, 1980. International Society of General relativity.

[9] A. E. Fischer and J. E. Marsden. The Einstein Evolution Equations as a First–Order Quasi–Linear Symmetric Hyperbolic System. *Communications in Mathematical Physics*, 28:1–38, May 1972.

[10] P. Gamblin. Solution réguliére á temps petit por l'équation d'euler–poisson. *Comm. Partial Differential Equations*, 18(5 & 6):731–745, 1993.

[11] Steven W. Hawking and G.R.W. Ellis. *The large scale structure of space–time.* Cambridge University Press, Cambridge, 1973.

[12] L. Hörmander. *Analysis of Linear Partial Differential Operators*, volume 1. Springer, New York, 1985.

[13] T. J.R. Hughes, T. Kato, and J. E. Marsden. Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity. *Arch. Rational Mech. Anal.*, 63(3):273–294 (1977), 1976.

[14] F. John. *Partial differential equations.* Springer, 1986.

[15] T. Kato. The Cauchy Problem for Quasi–Linear Symmetric Hyperbolic Systems. *Archive for Rational Mechanics and Analysis*, 58:181–205, 1975.

[16] T. Kato and G. Ponce. Commutator estimates and the Euler and Navier-Stokes equations. *Comm. Pure Appl. Math.*, 41(7):891–907, 1988.

[17] S.G. Krantz. *Partial Differential Equations and Complex Analysis.* Studies in Advanced Mathematics. CRC Press, Boca Raton, 1992.

[18] A. Majda. *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables.* Springer, New York, 1984.

[19] T. Makino. On a Local Existence Theorem for the Evolution Equation of Gaseous Stars. In T. Nishida, M. Mimura, and H. Fujii, editors, *Patterns and Waves*, pages 459–479, Amsterdam, 1986. North–Holland.

[20] Y. Meyer. *Regularite des solutions des equations aux derivatives patielles non lineares*. Lectures Notes in Mathematics 842. Springer-Verlag, 1984.

[21] L. Nirenberg and H. Walker. The null spaces of elliptic differential operators in $\mathbb{R}^n$. *Journal of Mathematical Analysis and Applications*, 42:271–301, 1973.

[22] T. A. Oliynyk. The newtonian limit for perfect fluids. *Commun. Math. Phys.*, 276:131-188, 2007.

[23] A. D. Rendall. The initial value problem for a class of general relativistic fluid bodies. *Journal of Mathematical Physics*, 33(2):1047–1053, October 1992.

[24] E.M. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton Mathematical Series No. 30. Princeton University Press, Princeton, N.J., 1970.

[25] M. E. Taylor. *Partial differential equations.Vol III, nonlinear equations*. Springer, 1997.

[26] M.E. Taylor. Pseudodifferential operators and nonlinear PDE. In *Progress in Mathematics*, volume 100. Birkhäuser, Boston-Basel-Berlin, 1991. Second printing 1993.

[27] H. Triebel. Spaces of Kudrjavcev type I. Interpolation, embedding, and structure. *J. Math. Anal. Appl.*, 56(2):253–277, 1976.

[28] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators*. Johann Ambrosis Barth, 1995.